A note on the monotone complexity of 2-REF

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2 December 1995

Abstract

Sharpening an argument in [2], we show that the problem of recognizing refutable propositional logic formulas in conjunctive normal form with two disjuncts is monotonic p-projection equivalent to the problem $st - DCON$ of directed connectivity between two distinguished vertices s, t of a directed graph. It follows by [4] that the monotonic depth of bounded fan-in boolean circuits for $2 - REF$ is $\Theta(\log^2(n))$.

Introduction

Let k-SAT [resp. k-REF] denote the set of satisfiable [resp. refutable[1]] propositional formulas in conjunctive normal form. It has long been known that 3-SAT is NP-complete and hence that 3-REF is $co - NP$-complete (S.A. Cook), while 2-REF is in NL hence computable in polynomial time. S.A. Cook and M. Luby gave a parallel algorithm (in NC^2) for the problem of testing whether a 2-CNF formula is satisfiable and if so yields a satisfying assignment. Appropriately formulated, 2-REF is a monotone problem; e.g.

$$(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{x} \vee y) \wedge (\bar{x} \vee \bar{y})$$

is refutable, so any extension

$$(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{x} \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (u \vee v)$$

is also refutable. In [2],

Let st-DCON denote the problem whether there is a path from the vertex s to vertex t in the directed graph G. This problem is monotone as well, since adding edges to G cannot remove a path from s to t. By showing that st-DCON is monotonic p-projection equivalent to 2-REF , we can apply the elegant monotonic lower bound result of M. Karchmer and A. Wigderson [4] to show that the monotonic depth of bounded fan-in boolean circuits computing 2-REF is $\Theta(\log^2(n))$.

1 Definitions

Definitions The language 2-REF of 2 conjunctive normal form refutable formulas is the problem of recognizing whether a formula ϕ of propositional logic is refutable,[2] provided that ϕ is in conjunctive normal form with 2 literals per disjunction. Formally, for $x = x_{|x|-1} \cdots x_0 \in \{0,1\}^*$, we make the following definition.

*Part of this research supported by NSF CCR-9408090 and US-Czech Science and Technology Grant 93-025.
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Key words and phrases: boolean circuit, depth, monotone circuit, projection reduction.

[1] ϕ is refutable iff ϕ is unsatisfiable iff $\neg\phi$ is provable.

[2] ϕ is refutable iff $\neg\phi$ is a tautology, iff ϕ is not satisfiable.

Definition 1

$$\text{2-REF} = \{x \in \{0,1\}^* : (\exists n \leq |x|)(|x| = 4n^2 \wedge \phi_{x,n} \textit{ is refutable})\}$$

where $\phi_{x,n}$ *is the formula* $A_{x,n} \wedge B_{x,n} \wedge C_{x,n} \wedge D_{x,n}$*, and*

$$\begin{aligned} A_{x,n} &= \bigwedge_{i,j<n, x_{ni+j}=1} p_i \vee p_j \\ B_{x,n} &= \bigwedge_{i,j<n, x_{n^2+ni+j}=1} \overline{p}_i \vee p_j \\ C_{x,n} &= \bigwedge_{i,j<n, x_{2n^2+ni+j}=1} p_i \vee \overline{p}_j \\ D_{x,n} &= \bigwedge_{i,j<n, x_{3n^2+ni+j}=1} \overline{p}_i \vee \overline{p}_j. \end{aligned}$$

The language st-DCON of $s-t$ directed connectivity is the problem, given a directed graph G and two distinct vertices s, t, whether there is a directed path from s to t. Formally, we make the following definition.

Definition 2

$$\textit{st}\text{-DCON} = \{x \in \{0,1\}^* : (\exists n \leq |x|)(|x| = n^2 \wedge \phi'_{x,n})\}$$

where $\phi'_{x,n}$ *expresses "there is a directed path from the vertex* 0 *to the vertex* $n-1$ *in the digraph* $G_{x,n} = (V_{x,n}, E_{x,n})$*", where* $V_{x,n} = \{0, \ldots, n-1\}$ *and* $E_{x,n} = \{(i,j) : i, j < n, x_{ni+j} = 1\}$.

For languages $L, L' \subseteq \{0,1\}^*$, L is p-projection reducible to L' (or L is a p-projection of L'), denoted $L \leq_{p-proj} L'$, iff there is a function p bounded by a polynomial with the property that

$$(\forall N)(\exists \sigma_N : \{y_i : i < p(N)\} \rightarrow (\{x_i, \overline{x}_i : i < N\} \cup \{0,1\}))$$

such that

$$(\forall x \in \{0,1\}^N)[x \in L \iff (\sigma_N(y_{p(N)-1}), \ldots, \sigma_N(y_0)) \in L'].$$

For $L, L' \subseteq \{0,1\}^*$, L is monotonic p-projection reducible to L' (or L is a monotonic p-projection of L'), denoted by $L \leq_{mp-proj} L'$, iff the function σ_N in the previous definition satisfies

$$\sigma_N : \{y_i : i < p(N)\} \rightarrow (\{x_i : i < N\} \cup \{0,1\}).$$

We write $L \equiv_{mp-proj} L'$ iff $L \leq_{mp-proj} L'$ and $L' \leq_{mp-proj} L$. The notion of p-projection was first introduced by L. Valiant in [7]. Both the notions of p-projection reduction and monotonic p-projection reduction are investigated by S. Skyum and L. Valiant in [8].

2 st-DCON is equivalent to 2-REF

The idea is to associate formula ϕ_G to graph G, where ϕ_G is a conjunction of s, $\bar{t}$, with the disjunctions $\bar{p}_i \vee p_j$ for each edge. Clearly if there is a path from s to t then ϕ_G cannot be satisfiable.

Theorem 3 *st*-DCON $\leq_{mp-proj}$2-REF .

Proof Let $p(N) = 4N$. Define

$$\sigma_N : \{y_m : m < p(N)\} \to (\{x_m : m < N\} \cup \{0,1\})$$

by the following.

Case 1. N is a perfect square.
Let $n = \sqrt{N}$. For $i, j < n$ and $k < 4$, we let

$$\sigma_N(y_{kn^2+ni+j}) = \begin{cases} x_{ni+j} & \text{if } k = 1 \\ 1 & \text{if } k = 0 \text{ and } i = 0 = j \\ 1 & \text{if } k = 3 \text{ and } i = n-1 = j \\ 0 & \text{else.} \end{cases}$$

Case 2. N is not a perfect square.
Let $\sigma_N(y_m) = 0$ for all $m < p(N)$.

Now, given $x = x_{N-1} \cdots x_0 \in \{0,1\}^*$, and $y = y_{p(N)-1} \cdots y_0$, we write $\sigma(y)$ instead of $(\sigma_N(y_{p(N)-1}), \ldots, \sigma_N(y_0))$. Clearly the formula $\phi_{\sigma(y),n}$ corresponding to definition 1 is given by the following.

Case 1. N is a perfect square.

$$\phi_{y,n} = (\bigwedge_{(i,j) \in G_{x,n}} (\overline{p}_i \vee p_j)) \wedge (p_0 \vee p_0) \wedge (\overline{p}_{n-1} \vee \overline{p}_{n-1}).$$

Case 2. N is not a perfect square.
In this case we have $\phi_{y,n} = \Box$, where $\Box$ represents the empty formula (a widely used convention in resolution).

The following well-known fact (see for instance Theorem 4 of [5]) completes this direction of the proof of the theorem.

Fact. If $(\exists n \leq x)(|x| = n^2 \wedge |y| = 4n^2)$ then there exists a directed path from 0 to $n-1$ in $G_{x,n}$ iff $\phi_{\sigma(y),n}$ is not satisfiable.

For $x \in \{0,1\}^*$ of length N, we now have the following.

Case 1. N is a perfect square.
In this case,

$$x \in \text{st-DCON} \iff (\sigma_N(y_{p(N)-1}), \ldots, \sigma_N(y_0)) \in \text{2-REF} .$$

Case 2. N is not a perfect square.
In this case, $x \notin$ st-DCON and $y \notin$ 2-REF . This establishes the theorem. ∎

Theorem 4 2-REF $\leq_{mp-proj}$ *st*-DCON .

Proof The idea is to associate graph G_ϕ for formula ϕ, where G_ϕ includes edges $(\bar{\alpha}, \beta)$ and $(\bar{\beta}, \alpha)$ for every disjunct $\alpha \vee \beta$ occurring in ϕ, where α,β are literals. To verify that this reduction can be done via monotonic p-reductions requires some work.

Let $p(N) = 36N$. If $N = 4n^2$ then boolean variables $x_0, \ldots, x_{N-1}$ encode a candidate formula possibly belonging to 2-REF . Before giving the formal details of the monotonic projection reduction, we give an intuitive description together with several figures. The proof breaks into two cases, the nontrivial case being when there exists an integer n for which $N = 4n^2$, so that $n = \frac{\sqrt{N}}{2}$. All of the intuitive discussion before the formal proof concerns only this nontrivial case.

In figure 1, the directed graph H_ϕ is presented with the property that H_ϕ is st-connected iff ϕ is refutable. The construction of H_ϕ is explained below.

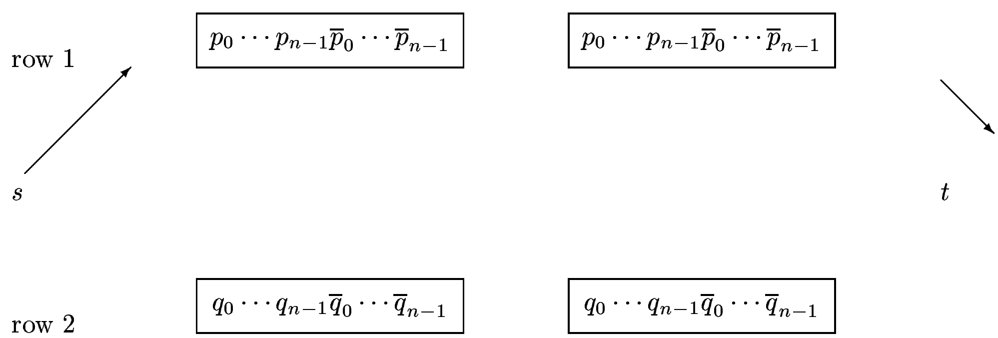

FIGURE 1

For ϕ in 2-CNF form with propositional variables $p_0, \ldots, p_{n-1}$, let G^1_ϕ be the graph obtained by taking vertex set $\{p_i, \overline{p}_i : i < n\}$ and with edge set given by the union of the following 4 sets

$$\begin{aligned}
&\{(\overline{p}_i, p_j) : \ p_i \vee p_j \text{ is a subformula of } \phi\}\\
&\{(\overline{p}_i, \overline{p}_j) : \ p_i \vee \overline{p}_j \text{ is a subformula of } \phi\}\\
&\{(p_i, p_j) : \ \overline{p}_i \vee p_j \text{ is a subformula of } \phi\}\\
&\{(p_i, \overline{p}_j) : \ \overline{p}_i \vee \overline{p}_j \text{ is a subformula of } \phi\}
\end{aligned}$$

Similarly, let G^2_ϕ be the graph obtained by taking vertex set $\{q_i, \overline{q}_i : i < n\}$ and with edge set given by the union of the following 4 sets

$$\begin{aligned}
&\{(\overline{q}_j, q_i) : \ p_i \vee p_j \text{ is a subformula of } \phi\}\\
&\{(q_j, q_i) : \ p_i \vee \overline{p}_j \text{ is a subformula of } \phi\}\\
&\{(\overline{q}_j, \overline{q}_i) : \ \overline{p}_i \vee p_j \text{ is a subformula of } \phi\}\\
&\{(q_j, \overline{q}_i) : \ \overline{p}_i \vee \overline{p}_j \text{ is a subformula of } \phi\}
\end{aligned}$$

Now define G_ϕ to be the union of the directed graphs G^1_ϕ and G^2_ϕ, together with edges (p_i, q_i), (q_i, p_i), $(\overline{p}_i, \overline{q}_i)$ $(\overline{q}_i, \overline{p}_i)$, for all $i < n$. We will now verify that ϕ is refutable if and only if for some $i < n$ there is a path from p_i to $\overline{p}_i$ and from $\overline{p}_i$ to p_i. (If such is the case, then $p_i, \overline{p}_i$ are said to be strongly connected.)

If ϕ is a formula in 2-CNF form and a, b are literals, then we define $Path_\phi(a, b)$ to hold iff there is a sequence $(a_0, a_1, \ldots, a_r)$ such that $a = a_0$, $b = a_r$, and for each $i < r$ either $a_i = a_{i+1}$ or $\overline{a}_i \vee a_{i+1}$ is a disjunct of ϕ or $a_{i+1} \vee \overline{a}_i$ is a disjunct of ϕ. In this case, r is said to be the length of the path $(a_0, \ldots, a_r)$. Note that, if for example $p_i \vee p_j$ is a disjunct of ϕ, then there is an edge $(\overline{p}_i, p_j)$ in G^1_ϕ and an edge $(\overline{q}_j, q_i)$ in G^2_ϕ. Since there are edges between p_k and q_k and between $\overline{p}_k$ and $\overline{q}_k$, for all $k < n$, there is a path from $\overline{p}_i$ to p_j and from $\overline{p}_j$ to p_i in G_ϕ. As well, note that we identify $\overline{\overline{a}}_i$ with a_i. The proof of the following is well known and can be found, for instance, in Theorem 9.4 on page 184 of [6].

Fact. If ϕ is a formula in 2-CNF form, then ϕ is refutable iff

$$(\exists i < n)(Path_\phi(p_i, \overline{p}_i) \wedge Path_\phi(\overline{p}_i, p_i)).$$

Returning to our intuitive description, given $x_0, \ldots, x_{N-1}$, where $N = 4n^2$, which possibly encodes a refutable formula in 2-CNF, we define the graph H_ϕ to have vertex set $\{0, \ldots, 12n-1\}$, where nodes $8n+1, \ldots, 12n-2$ are isolated (i.e. not incident to any edge), and hence not drawn in the figures. Graph H_ϕ is encoded by the boolean variables $y_0, \ldots, y_{p(N)-1}$, where $p(N) = 36N = (12n)^2$, which is greater than $(8n+2)^2$ for $n \geq 1$. H_ϕ has 2 designated vertices called s and t (in the encoding, s corresponds to 0 and t to $12n-1$) together with two copies of the graph G_ϕ. We think of each copy of G_ϕ as arranged in two rows, with G^1_ϕ in the top row and G^2_ϕ in the bottom row. Thus we consider the vertices of H_ϕ as arranged in two rows, where in the first row there are two copies of G^1_ϕ, and in the second row there are two copies of G^2_ϕ. In addition to edges from G_ϕ, there are edges from each of the $\overline{p}_i$ vertices in the first copy of G^1_ϕ to the corresponding $\overline{p}_i$ vertex in the second copy of G^1_ϕ. As well, there are edges from each of the $\overline{q}_i$ vertices in the first copy of G^2_ϕ to the corresponding $\overline{q}_i$ vertex in the second copy of G^2_ϕ. Apart from these edges, H_ϕ has additional s-edges and t-edges. Namely, the vertex s is joined to p_k in the first copy of G^1_ϕ in the first row, for $k < n$. For $k < n$, each p_k in the second copy of G^1_ϕ in the first row is joined to t. We think of enumerating the vertices of H_ϕ, beginning with 0 (the s vertex), then with the vertices $1, \ldots, 2n$ of the first copy of G^1_ϕ, continuing with $2n+1, \ldots, 4n$ in the second copy of G^1_ϕ, continuing with $4n+1, \ldots, 6n$ for the first copy of G^2_ϕ, continuing with $6n+1, \ldots, 8n$ for the second copy of G^2_ϕ, followed by $8n+1$ (the t vertex). Within a copy of G^1_ϕ, for instance with vertex set $1, \ldots, 2n$, we think of vertices $1, \ldots, n$ [resp. $n+1, \ldots, 2n$] corresponding to $p_0, \ldots, p_{n-1}$ [resp. $\overline{p}_0, \ldots, \overline{p}_{n-1}$]. Similarly for the other copy of G^1_ϕ and for both copies of G^2_ϕ.

We have already shown that ϕ is refutable iff there exists $i < n$ for which $p_i, \overline{p}_i$ are strongly connected. The latter clearly holds iff H_ϕ is st-connected.

The adjacency matrix of H_ϕ then has $p(N) = 36N = (12n)^2 \geq (8n+2)^2$ many entries, which is encoded by boolean variables $y_0, \ldots, y_{p(N)-1}$, whose formal definition is given later.

Namely, the intent of the definition of boolean variables $y_0, \ldots, y_{36N-1}$ is as follows. The first clause defines all "s-edges". The second clause defines all "t-edges". The third clause defines those edges from $\overline{p}_k$ in the first copy of G^1_ϕ to $\overline{p}_k$ in the second copy of G^1_ϕ in both rows, for all $k < n$. The fourth clause defines edges from p_k to q_k, from q_k to p_k, from $\overline{p}_k$ to $\overline{q}_k$, and from $\overline{q}_k$ to $\overline{p}_k$ for all $k < n$ in both copies of G^1_ϕ and G^2_ϕ. The fifth clause defines edges corresponding to G^1_ϕ in both copies of G^1_ϕ on the first row. The sixth clause defines edges corresponding to G^2_ϕ in both copies of G^2_ϕ on the second row. Here note that $\lfloor \frac{i-1}{2n} \rfloor = \lfloor \frac{j-1}{2n} \rfloor$ holds exactly when i, j are vertices of H_ϕ distinct from s, t, and which are in the same G^1_ϕ copy or same G^2_ϕ copy.

Formal details now follow.

Case 1. There exists an integer n for which $N = 4n^2$.
In this case, let $n = \frac{\sqrt{N}}{2}$ so that $N = 4 \cdot n^2$.

Define $u, v \in \{0,1\}^N$ for $i, j < 2n$ by

$$u_{i(2n)+j} = \begin{cases} x_{1 \cdot n^2+ni+j} & \text{if } i,j < n \\ x_{3 \cdot n^2+ni+(j-n)} & \text{if } i < n \text{ and } n \leq j < 2n \\ x_{0 \cdot n^2+n(i-n)+j} & \text{if } n \leq i < 2n \text{ and } j < n \\ x_{2 \cdot n^2+n(i-n)+(j-n)} & \text{if } n \leq i,j < 2n \end{cases}$$

and

$$v_{i(2n)+j} = \begin{cases} x_{2 \cdot n^2+nj+i} & \text{if } i,j < n \\ x_{3 \cdot n^2+n(j-n)+i} & \text{if } i < n \text{ and } n \leq j < 2n \\ x_{0 \cdot n^2+nj+(i-n)} & \text{if } n \leq i < 2n \text{ and } j < n \\ x_{1 \cdot n^2+n(j-n)+(i-n)} & \text{if } n \leq i,j < 2n. \end{cases}$$

Referring to the notation of Definition 2, the following equivalences for $i, j < 2n$ and $G_{u,2n} = (V_{u,2n}, E_{u,2n})$, are immediately derived from the above definitions of the boolean variables $u_0, \ldots, u_{N-1}$ and $v_0, \ldots, v_{N-1}$.

- For $i, j < n$

$$\begin{aligned} (i,j) \in E_{u,2n} &\iff u_{i(2n)+j} = 1 \\ &\iff x_{n^2+ni+j} = 1 \\ &\iff \overline{p}_i \vee p_j \text{ is a subformula of } \phi_{x,n} \end{aligned}$$

- For $i < n$, $n \leq j < 2n$

$$\begin{aligned} (i,j) \in E_{u,2n} &\iff u_{i(2n)+j} = 1 \\ &\iff x_{3n^2+ni+(j-n)} = 1 \\ &\iff \overline{p}_i \vee \overline{p}_{j-n} \text{ is a subformula of } \phi_{x,n} \end{aligned}$$

- For $n \leq i < 2n$, $j < n$

$$\begin{aligned} (i,j) \in E_{u,2n} &\iff u_{i(2n)+j} = 1 \\ &\iff x_{n(i-n)+j} = 1 \\ &\iff p_{i-n} \vee p_j \text{ is a subformula of } \phi_{x,n} \end{aligned}$$

- For $n \leq i, j < 2n$

$$\begin{aligned} (i,j) \in E_{u,2n} &\iff u_{i(2n)+j} = 1 \\ &\iff x_{2n^2+n(i-n)+(j-n)} = 1 \\ &\iff p_{i-n} \vee \overline{p}_{j-n} \text{ is a subformula of } \phi_{x,n} \end{aligned}$$

As well, for $i, j < 2n$ and $G_{v,2n} = (V_{v,2n}, E_{v,2n})$, the following is similarly derived.

- For $i, j < n$

$$\begin{aligned} (i,j) \in E_{v,2n} &\iff v_{i(2n)+j} = 1 \\ &\iff x_{2n^2+nj+i} = 1 \\ &\iff p_j \vee \overline{p}_i \text{ is a subformula of } \phi_{x,n} \end{aligned}$$

- For $i < n$, $n \leq j < 2n$

$$\begin{aligned} (i,j) \in E_{v,2n} &\iff v_{i(2n)+j} = 1 \\ &\iff x_{3n^2+n(j-n)+i} = 1 \\ &\iff \overline{p}_{j-n} \vee \overline{p}_i \text{ is a subformula of } \phi_{x,n} \end{aligned}$$

- For $n \leq i < 2n$, $j < n$

$$
\begin{aligned}
(i,j) \in E_{v,2n} &\iff v_{i(2n)+j} = 1 \\
&\iff x_{jn+(i-n)} = 1 \\
&\iff p_j \vee p_{i-n} \text{ is a subformula of } \phi_{x,n}
\end{aligned}
$$

- For $n \leq i, j < 2n$

$$
\begin{aligned}
(i,j) \in E_{v,2n} &\iff v_{i(2n)+j} = 1 \\
&\iff x_{n^2+n(j-n)+(i-n)} = 1 \\
&\iff \overline{p}_{j-n} \vee p_{i-n} \text{ is a subformula of } \phi_{x,n}
\end{aligned}
$$

In the current case, $|x| = 4n^2 = N$ and we are to define y satisfying $|y| = p(N) = 36N$. Note that $(8n+2)^2 \leq (12n)^2 = 36N$, for $n \geq 1$. For $i, j < 12n$, define

$$
y_{i(12n)+j} = \begin{cases}
1 & \text{if } i = 0 \text{ and } 1 \leq j \leq n \\
1 & \text{if } j = 12n-1 \text{ and } 2n+1 \leq i \leq 3n \\
1 & \text{if } j = i + 2n \text{ and} \\
 & \text{ either } (n+1 \leq i \leq 2n) \\
 & \text{ or } (5n+1 \leq i \leq 6n) \\
1 & \text{if } (j = i + 4n \wedge 1 \leq i \leq 4n) \\
 & \text{or } (i = j + 4n \wedge 1 \leq j \leq 4n) \\
u_{((i-1) \bmod 2n)\cdot 2n + (j-1) \bmod 2n} & \text{if } 1 \leq i, j \leq 4n \text{ and} \\
 & \lfloor \frac{i-1}{2n} \rfloor = \lfloor \frac{j-1}{2n} \rfloor \\
v_{((i-1) \bmod 2n)\cdot 2n + (j-1) \bmod 2n} & \text{if } 4n+1 \leq i, j \leq 8n \text{ and} \\
 & \lfloor \frac{i-1}{2n} \rfloor = \lfloor \frac{j-1}{2n} \rfloor \\
0 & \text{if none of the above cases holds.}
\end{cases}
$$

Case 2. There is no integer n for which $N = 4n^2$.
In this case, for all $i, j < 12n$ define $y_{i(12n)+j} = 0$.

From the above cases, it is straightforward to define the corresponding function

$$\sigma_N : \{y_i : i < p(N)\} \to (\{x_i : i < N\} \cup \{0,1\}).$$

It now follows from the previous fact and the definition of σ_N that

$$(\forall x \in \{0,1\}^N)(x \in 2\text{-REF} \iff (\sigma_N(y_{p(N)-1}), \ldots, \sigma_N(y_0)) \in \text{st-DCON} .$$

This completes the proof of the theorem. ∎

Putting the previous two results together, we have the following.

Theorem 5 2-REF $\equiv_{mp-proj}$ *st*-DCON .

For $L \subseteq \{0,1\}^*$, we denote $L \cap \{0,1\}^N$ by L_N. Let $d_m(L_N)$ denote the least depth of a monotonic boolean circuit with AND, OR gates of fan-in 2, no NOT gates, where circuit inputs are among the constants 0, 1 and the positive literals $x_0, \ldots, x_{N-1}$. If $f : \mathbf{N} \to \mathbf{N}$, then we write $d_m(L) = \Theta(f)$ to mean $d_m(L_N) = \Theta(f(N))$.

Corollary 6 $d_m(2\text{-REF}\ ) = \Theta(\log^2)$.

Proof By the previous theorem, 2-REF $\equiv_{mp-proj}$ st-DCON . By the elegant result of Karchmer-Wigderson [4], $d_m(\text{st-DCON}\ ) = \Theta(\log^2 n)$. Clearly, monotone p-projections preserve monotonic circuit bounds, and so $d_m(2\text{-REF}\ ) = \Theta(\log^2 n)$. ∎

A small example might provide an illustration of the encoding in theorem 4. Suppose that ϕ is the following refutable formula

$$(\overline{p}_0 \vee p_1) \wedge (p_0 \vee \overline{p}_1) \wedge (\overline{p}_0 \vee \overline{p}_1) \wedge (p_0 \vee p_1).$$

Then the graph H_ϕ is depicted below. As well, a directed path from s to t is given by

$$s, p_0, \overline{p}_1, \overline{q}_1, \overline{q}_1, \overline{q}_0, \overline{p}_0, p_1, t.$$

In this example, $n = 2$, $N = 4n^2 = 16$, $p(N) = 36N = 576$, and $(8n + 2)^2 = 324 \leq (12n)^2 = 36N = 576$. The encoding of the graph H_ϕ on vertex set $\{0, \ldots, 23\}$ is depicted below. Isolated points $8n+1, \ldots, 12n-2$ (or $17, \ldots, 22$) are not depicted. As well, a directed path from the s vertex (encoded by 0) to the t vertex (encoded by 23) is given by

$$0, 1, 4, 12, 16, 15, 7, 6, 23.$$

3 Acknowledgements

I would like to give special thanks to Pierre McKenzie for pointing out a number of typographic errors and especially for finding and correcting an error in the encoding used in an earlier draft. Thanks also to R. Boppana and L. Valiant for providing certain references.

References

[1] R. Boppana, M. Sipser, "The complexity of finite functions", pp. 757-804, *Handbook of Theoretical Computer Science*, Volume A - Algorithms and Complexity, North-Holland and MIT Press (1990).

[2] S.A. Cook, M. Luby, "A simple parallel algorithm for finding a satisfying truth assignment to a 2-CNF formula" *IPL*, **27** (1988) 141-145.

[3] S.A. Cook, "A taxonomy of problems with fast parallel algorithms", *Information and Control* **64** (1985) 2-22.

[4] M. Karchmer, A. Wigderson, "Monotone circuits for connectivity require super-logarithmic depth", *J. Discr. Math.* **3** (1990) pp 255–265. Preliminary extended abstract appeared in *Proc. 20th Ann. ACM Symp. on Theory of Computing* (1988) 539-550.

[5] N.D. Jones, Y.E. Lien and W.T. Laaser, "New problems complete for nondeterministic log space", *Math. Sys. Theory* **10** (1976) pp 1–17.

[6] C. Papadimitriou, *Computational Complexity*, Addison-Wesley (1994) 523 pages.

[7] L. Valiant, "Completeness classes in algebra", *Proceedings of the 11th Annual ACM Symposium on Theory of Computing* 249-261 (1979).

[8] S. Skyum, L. Valiant, "A complexity theory based on boolean algebra", *JACM* **32** (2), 484-502 (1985).