

# On a generalization of $PHP$ , $st$ -connectivity, and odd charged graphs

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## 1 Introduction

As witnessed by this proceedings, there is currently much interest in the analysis of proof size for tautology families in certain proof systems for propositional logic. The reason is twofold: (1) the analysis of proof systems leads to a better understanding of efficiency issues for theorem provers and (2) the development of new combinatorial methods in establishing proof size lower bounds for propositional proof systems may help in better understanding and solving difficult problems in complexity theory (it is well-known that  $NP = co-NP$  if and only if there is a sound, complete propositional proof system which furnishes polynomial size proofs for all tautologies).

In this paper, we consider several proof systems (resolution, cutting planes, and a multiplicative extension of cutting planes), and analyze the proof size of certain combinatorial statements related to the pigeonhole principle and to graph theoretic principles.

The well-known pigeonhole principle  $PHP_k$  is given by

$$\bigwedge_{0 \leq i \leq k} \bigvee_{0 \leq j < k} p_{i,j} \rightarrow \bigvee_{0 \leq i < i' \leq k} \bigvee_{0 \leq j < k} (p_{i,j} \wedge p_{i',j}).$$

In [7], W. Degen gave a natural generalization of the pigeonhole principle, which states that for positive integers  $m, k$  if  $f$  is a function mapping  $\{0, \dots, m \cdot k\}$  into  $\{0, \dots, k-1\}$  then there is  $j < k$  for which  $f^{-1}(j)$  has size greater than  $m$ . For ease of notation, non-negative integers will be considered as von Neumann ordinals, so that  $n = \{0, \dots, n-1\}$ , and  $[m]^n$  denotes the set of size  $n$  subsets

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of  $\{0, \dots, m-1\}$ . Formulated in propositional logic, Degen's generalization is given by a family  $\{D_{m,k} : m, k \in \mathbf{N} - \{0\}\}$  where  $D_{m,k}$  is

$$\bigwedge_{0 \leq i \leq m \cdot k} \bigvee_{0 \leq j < k} p_{i,j} \rightarrow \bigvee_{0 \leq j < k} \bigvee_{I \in [m \cdot k + 1]^{m+1}} \bigwedge_{i \in I} p_{i,j}.$$

W. Degen showed that for  $m$  fixed, over  $ZF$  set theory without the axiom of choice, the set theoretic analogue of  $\{D_{m,k} : k \in \mathbf{N}\}$  is properly weaker than the set theoretic analogue of  $\{D_{m+1,k} : k \in \mathbf{N}\}$  when  $m+1$  is a prime. The pigeonhole principle, Ajtai's parity principle, and various modular counting principles have been investigated in boolean circuit complexity and in propositional proof theory, the idea being that counting is a difficult notion to capture in finite depth circuits or proofs. Motivated by Degen's surprising hierarchy result in set theory, we investigated his principle in propositional logic. This paper establishes that Degen's principle is of the same strength as the pigeonhole principle. Additionally, we consider a propositional logic formulation of *st*-connectivity, and using the Karchmer-Wigderson lower bound for monotonic circuits, furnish an example where tree-like resolution is weaker than resolution (by a different proof, a separation between tree-like resolution and resolution was first given by Tseitin). Finally, we note the similarity of operations allowed in defining *graph minors* and those in restricted resolution proofs. This raises the question of a graph theoretic characterization of those bounded degree graphs whose Tseitin tautologies require large proofs in resolution or restricted resolution. In this context, we prove a relation between the pigeonhole principle and Tseitin's odd-charged graphs.

## 2 Preliminaries

We refer the reader to Krajíček's book [11] for any undefined terminology. Resolution is a sound, complete refutation system for conjunctive normal form (*CNF*) formulas – *sound*, in that if *CNF* formula  $\phi$  has a resolution refutation, then  $\phi$  is unsatisfiable, and *complete*, in that every unsatisfiable *CNF* formula has a resolution refutation. *CNF* formulas are represented in resolution by a set of clauses containing literals (a literal is a propositional variable or its negation), where the clause  $\{\alpha_1, \dots, \alpha_n\}$  represents  $\alpha_1 \vee \dots \vee \alpha_n$ . The resolution rule allows the derivation of clause  $C \cup D$  from the clauses  $C \cup \{x\}$  and  $D \cup \{\bar{x}\}$ . A resolution derivation from  $C_1, \dots, C_n$  is a sequence of clauses  $D_1, \dots, D_m$ , such that every  $D_i$  is either one of the  $C$ 's, or obtained from  $D_j, D_k$  for  $j, k < i$  by the resolution rule. A resolution refutation of clauses  $C_1, \dots, C_n$  is a derivation of the empty clause from  $C_1, \dots, C_n$ . By abuse of notation, we say that a disjunctive normal form formula has a resolution proof, if its negation (a *CNF* formula) has a resolution refutation. A resolution derivation is tree-like if every clause is used at most once in an application of the resolution rule (multiple resolutions on the same clause require multiple derivations of that clause).

The cutting plane proof system,  $CP$ , is a sound and complete refutation system for  $CNF$  formulas. Propositional variable  $x_i$  is represented by itself;  $\neg x_i$  is represented by  $1 - x_i$ ; a disjunction  $\bigvee_{i \in I} \alpha_i$  of literals is represented by  $\sum_{i \in I} R(\alpha_i) \geq 1$ , where  $R(\alpha_i)$  represents the literal  $\alpha_i$ ; finally, a  $CNF$  formula  $\bigwedge_{i \in I} \bigvee_{j \in J_i} \alpha_{i,j}$  is represented by the family

$$\sum_{j \in J_i} R(\alpha_{i,j}) \geq 1$$

of linear inequalities. Without loss of generality, we assume all linear inequalities are of the form  $\sum a_i \cdot x_i \geq A$  where  $a_i, A \in \mathbf{Z}$ . The  $a_i$  are the coefficients of the propositional variables, and for lack of a better term, we call  $A$  the integer sum. The *axioms* of  $CP$  are  $x_i \geq 0$ ,  $-x_i \geq -1$ . The *rules of inference* of  $CP$  are

- addition  $\frac{\sum a_i \cdot x_i \geq A \quad \sum b_i \cdot x_i \geq B}{\sum (a_i + b_i) \cdot x_i \geq A + B}$

- division  $\frac{\sum (c \cdot a_i) \cdot x_i \geq A}{\sum a_i \cdot x_i \geq \lceil \frac{A}{c} \rceil}$

where integer  $c > 1$ ,

- multiplication  $\frac{\sum a_i \cdot x_i \geq A}{\sum (c \cdot a_i) \cdot x_i \geq c \cdot A}$

where integer  $c > 1$ .

A derivation  $D$  for inequalities  $I$  from inequalities  $I_1, \dots, I_m$  is a sequence  $D = (D_0, \dots, D_n)$  such that for all  $i \leq n$  either  $D_i$  is an axiom, or one of  $I_1, \dots, I_m$  or inferred from  $D_j, D_k$  for  $j, k < i$  by means of a rule of inference. A *refutation* of  $I_1, \dots, I_m$  is a derivation of  $0 \geq 1$  from  $I_1, \dots, I_m$ . As in the case of resolution, by abuse of terminology, we say that a disjunctive normal form formula has a  $CP$  proof if its negation has a  $CP$  refutation. The *size* of a  $CP$  refutation is the sum over all inequalities  $\sum a_i \cdot x_i \geq A$  occurring in the refutation of  $\sum |a_i| + |A|$ , where  $|A|$  indicates the length of the binary representation of  $A$ . It is easy to see [6] that  $CP$  is a sound extension of resolution, hence complete.

### 3 Cutting plane proofs of Degen's principle

By  $E_{m,k}$  we denote the  $CP$  inequalities corresponding to the  $CNF$  formula  $\neg D_{m,k}$ . Thus  $E_{m,k}$  is

$$\sum_{j=0}^{k-1} p_{i,j} \geq 1$$

for  $0 \leq i \leq m \cdot k$ , together with

$$-p_{i_1,j} - p_{i_2,j} - \cdots - p_{i_{m+1},j} \geq -m$$

for  $0 \leq j < k$  and  $0 \leq i_1 < i_2 < \cdots < i_{m+1} \leq m \cdot k$ .

**Theorem 1** *There are  $O(k^5)$  size CP refutations of  $E_{2,k}$ .*

**Proof** By assumption from  $E_{2,k}$ , for all  $0 \leq i_1 < i_2 < i_3 \leq 2k$  and all  $0 \leq r < k$ ,

$$2 \geq p_{i_1,r} + p_{i_2,r} + p_{i_3,r}.$$

**Claim 2** *For all  $0 \leq i_1 < i_2 < i_3 < i_4 \leq 2k$  and all  $0 \leq r < k$ ,*

$$2 \geq p_{i_1,r} + p_{i_2,r} + p_{i_3,r} + p_{i_4,r}.$$

**Proof of claim:** Fix  $i_1, i_2, i_3, i_4$  and  $r$ , and temporarily, set  $a = p_{i_1,r}$ ,  $b = p_{i_2,r}$ ,  $c = p_{i_3,r}$ ,  $d = p_{i_4,r}$ . By assumption from  $E_{2,k}$ , we have

$$\begin{aligned} 2 &\geq a + b + c \\ 2 &\geq b + c + d \\ 2 &\geq a + b + d \\ 2 &\geq a + c + d \end{aligned}$$

and so by addition

$$8 \geq 3a + 3b + 3c + 3d$$

and hence by division by 3

$$2 = \lceil 8/3 \rceil \geq a + b + c + d.$$

□

For later generalization, note that the pattern of the previous inequalities is of the following form:

$$\begin{array}{cccc} + & + & + & - \\ - & + & + & + \\ + & - & + & + \\ + & + & - & + \end{array}$$

where + [resp. -] indicates presence [resp. absence] of the corresponding element (i.e. in the first row, there is  $a, b, c$  but no  $d$  present). In this manner, with  $O(k^5)$  (i.e. order  $k \cdot \binom{2k+1}{4}$ ) many proof lines we can show that

$$2 \geq p_{i_1,r} + \cdots + p_{i_4,r}$$

for all rows  $0 \leq r < k$  and all 4-tuples  $0 \leq i_1 < i_2 < i_3 < i_4 \leq 2 \cdot k$  from that row. In a similar manner, we could show by a proof of  $O(k^{s+1})$  lines,

that  $2 \geq p_{i_1,r} + \cdots + p_{i_s,r}$ , for all rows  $0 \leq r < k$  and all distinct  $s$ -tuples  $i_1, \dots, i_s$ . However, the overall proof would then be of  $\sum_{i=5}^{2k+1} O(k^i)$  lines, hence of exponential size. For that reason, in the following claim, we consider sets  $i_1, \dots, i_s$  of a particular form. Define integers  $x_1, \dots, x_m$  to be *consecutive* if for all  $1 \leq j < m$ ,  $x_{j+1} = x_j + 1$ .

**Claim 3** *Assume that  $3 \leq s \leq 2k$  and for all  $0 \leq i_1 < \cdots < i_s \leq 2k$  such that  $i_2, \dots, i_s$  are consecutive, and for all  $0 \leq r < k$ , it is the case that*

$$2 \geq p_{i_1,r} + \cdots + p_{i_s,r}.$$

*Then for all  $0 \leq i_1 < \cdots < i_{s+1} \leq 2k$  such that  $i_2, \dots, i_{s+1}$  are consecutive, and for all  $0 \leq r < k$ , it is the case that*

$$2 \geq p_{i_1,r} + \cdots + p_{i_{s+1},r}.$$

**Proof of claim:** Fix  $0 \leq i_1 < \cdots < i_{s+1}$  and  $r$ . By assumption

$$\begin{aligned} 2 &\geq p_{i_1,r} + \cdots + p_{i_s,r} \\ 2 &\geq p_{i_2,r} + \cdots + p_{i_{s+1},r} \\ 2 &\geq p_{i_1,r} + p_{i_3,r} + \cdots + p_{i_{s+1},r} \\ 2 &\geq p_{i_1,r} + p_{i_2,r} + p_{i_{s+1},r} \end{aligned}$$

Note that the pattern in the previous inequalities is of the following form:

$$\begin{array}{cccccc} + & + & + & \cdots & + & - \\ - & + & + & \cdots & + & + \\ + & - & + & \cdots & + & + \\ + & + & - & \cdots & - & + \end{array}$$

The first three inequalities hold by the assumption in the claim, and the fourth (which contains only 3 terms) holds by assumption of  $E_{2,k}$ . By addition, we have

$$8 \geq 3p_{i_1,r} + \cdots + 3p_{i_{s+1},r}$$

and hence by division by 3

$$2 = \lceil 8/3 \rceil \geq p_{i_1,r} + \cdots + p_{i_{s+1},r}.$$

□

By induction on  $s$ , using the base case  $2 \geq p_{i_1,r} + p_{i_2,r} + p_{i_2+1,r}$  for all  $0 \leq r < k$  and  $0 \leq i_1 < i_2 \leq 2 \cdot k$  (given by  $E_{2,k}$ ), and applying Claim 3 in the inductive case, it follows that for all  $0 \leq r < k$ ,

$$2 \geq p_{0,r} + \cdots + p_{2k,r}.$$

Adding all  $k$  inequalities (one for each  $0 \leq r < k$ ), we have

$$2k \geq \sum_{i=0}^{2k} \sum_{j=0}^{k-1} p_{i,j}.$$

However, by hypothesis  $E_{2,k}$ , for each fixed  $0 \leq i \leq 2k$ ,  $\sum_{j=0}^{k-1} p_{i,j} \geq 1$ , and by addition of these  $2k + 1$  inequalities (one for each  $0 \leq i \leq 2k$ ), we have

$$\sum_{i=0}^{2k} \sum_{j=0}^{k-1} p_{i,j} \geq 2k + 1.$$

Thus we arrive at the contradiction  $2k \geq 2k + 1$ . Rewriting the above proof in the required normal form  $\sum a_{i,j} \cdot p_{i,j} \geq A$  we obtain a derivation of  $0 \geq 1$  from  $E_{2,k}$ .

What is the size of this  $CP$  refutation? In Claim 3, for each fixed  $s \geq 3$ , there are at most  $O(2k)$  choices of  $0 \leq i_1 \leq 2k$  and (by consecutivity) at most  $O(2k)$  choices of the remaining consecutive  $0 \leq i_2, \dots, i_{s+1} \leq 2k$  with  $i_1 < i_2$ . There are  $k$  many values of  $0 \leq r < k$ , so altogether this makes  $O(k^3)$  proof lines for establishing the claim in going from  $s$  to  $s + 1$ . As  $s \leq 2k$ , the entire proof requires  $O(k^4)$  lines. The coefficients of the propositional variables have size bounded by 2 (the largest coefficient is 3). Except in the last two steps, where at most  $2k + 1$  inequalities are added (producing sums  $2k$  and  $2k + 1$ ), all sums are bounded by 8. Each inequality (proof line) has size  $O(k)$ , since coefficients of the variables are bounded by a constant, the integer sum is bounded by  $2k + 1$ , and there are at most  $O(k)$  variables per inequality. Thus the proof size is  $O(k^4 \cdot k)$  or  $O(k^5)$ . ■

**Theorem 4** *Let  $m \geq 2$  and  $n = m \cdot k + 1$ . Then there are  $O(n^{m+3})$  size  $CP$  refutations of  $E_{m,k}$ , where the constant in the  $O$ -notation depends on  $m$ , and  $O(n^{m+4})$  size  $CP$  refutations, where the constant is independent of  $n, m$ .*

**Proof** We generalize the proof of the previous theorem.

**Claim 5** *Assume that  $3 \leq s \leq mk$  and for all  $0 \leq i_1 < \dots < i_s \leq mk$  such that  $i_m, \dots, i_s$  are consecutive, and for all  $0 \leq r < k$ , it is the case that*

$$m \geq p_{i_1,r} + \dots + p_{i_s,r}.$$

*Then for all  $0 \leq i_1 < \dots < i_{s+1} \leq mk$  such that  $i_m, \dots, i_{s+1}$  are consecutive, and for all  $0 \leq r < k$ , it is the case that*

$$m \geq p_{i_1,r} + \dots + p_{i_{s+1},r}.$$

**Proof of claim:** Fix  $i_1 < \dots < i_{s+1}$  and  $r$ . We have the following  $m + 2$  inequalities:

$$\begin{aligned}
m &\geq p_{i_1,r} + \dots + p_{i_s,r} \\
m &\geq p_{i_2,r} + \dots + p_{i_{s+1},r} \\
m &\geq p_{i_1,r} + p_{i_3,r} + \dots + p_{i_{s+1},r} \\
m &\geq p_{i_1,r} + p_{i_2,r} + p_{i_4,r} + \dots + p_{i_{s+1},r} \\
m &\geq p_{i_1,r} + \dots + p_{i_3,r} + p_{i_5,r} + \dots + p_{i_{s+1},r} \\
m &\geq p_{i_1,r} + \dots + p_{i_4,r} + p_{i_6,r} + \dots + p_{i_{s+1},r} \\
&\vdots \\
m &\geq p_{i_1,r} + \dots + p_{i_{m-1},r} + p_{i_{m+1},r} + \dots + p_{i_{s+1},r} \\
m &\geq p_{i_1,r} + \dots + p_{i_m,r} + p_{i_{s+1},r}
\end{aligned}$$

The pattern of terms in the  $m + 2$  inequalities above is of the form:

$$\begin{array}{cccccccc}
+ & + & + & \cdots & + & + & + & - \\
- & + & + & \cdots & + & + & + & + \\
+ & - & + & \cdots & + & + & + & + \\
+ & + & - & \cdots & + & + & + & + \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
+ & + & + & \cdots & + & - & + & + \\
+ & + & + & \cdots & + & + & - & -
\end{array}$$

Removal of any of the first  $m - 1$  summands in the term  $p_{i_1,r} + \dots + p_{i_{s+1},r}$  produces a term where  $p_{i_m,r}, \dots, p_{i_{s+1},r}$  are consecutive. This observation, with the assumption in the claim, justifies the first  $m + 1$  inequalities. The last inequality (which contains only  $m + 1$  terms) holds by assumption of  $E_{m,k}$ . By addition, we have

$$m \cdot (m + 2) \geq (m + 1) \cdot (p_{i_1,r} + \dots + p_{i_{s+1},r})$$

and hence by division by  $m + 1$

$$m = \lfloor \frac{m(m + 2)}{m + 1} \rfloor \geq p_{i_1,r} + \dots + p_{i_{s+1},r}.$$

□

Adding  $k$  inequalities  $m \geq p_{0,r} + \dots + p_{mk,r}$ , we have  $mk \geq \sum_{i=0}^{mk} \sum_{j=0}^{k-1} p_{i,j}$ . Similarly adding the  $mk + 1$  inequalities  $p_{i,0} + \dots + p_{i,k-1} \geq 1$ , we have  $\sum_{i=0}^{mk} \sum_{j=0}^{k-1} p_{i,j} \geq mk + 1$ . Finally, we have the desired contradiction  $mk \geq mk + 1$ .

Let  $n = mk$ . In Claim 5, for each fixed  $3 \leq s \leq mk$ , there are at most  $O(n^m)$  choices of  $0 \leq i_1 < \dots < i_s \leq mk$  for which  $i_m, \dots, i_s$  are consecutive (such sequences are stipulated by choice of  $i_1, \dots, i_m$ ). There are  $k$  many values of  $0 \leq r < k$ , so altogether this makes  $O(n^{m+1})$  proof lines for establishing the claim in going from  $s$  to  $s + 1$ . As  $s \leq mk$ , the entire proof requires  $O(n^{m+2})$  lines. In each proof line (inequality), there are at most  $mk + 1$  variables (hence  $O(k)$ , if  $m$  is held as a constant), while the coefficients of the variables are bounded by  $m + 1$ , and the integer sum (except in the last steps where  $k$  inequalities are added together) are bounded by  $m(m + 2)$ . In the last steps involving addition of  $k$  inequalities, the integer sum is bounded by  $mk + 1$ , hence the size of each inequality (proof line) in the proof is bounded by  $O(n)$  if  $m$  is held as a constant. It follows that the proof size is  $O(n^{m+2} \cdot n)$  or  $O(n^{m+3})$ , where the constant in the  $O$ -notation depends only on  $m$ . The previous considerations yield that the proof size is  $O(n^{m+3} \cdot \log m)$  or  $O(n^{m+4})$ , where the constant in the  $O$ -notation is an absolute constant independent of  $n, m$ . ■

**Corollary 6** *There are polynomial size CP refutations of  $\{E_{m,k} : m \geq 2, k \geq 1\}$ .*

**Proof** Each application of Claim 3 requires  $m + 1$  additions and one division. Taking this into account, as in the previous analysis, there are then  $O(m \cdot n^{m+3})$  proof lines, each of size  $O(\log m)$ , so the refutation size of  $E_{m,k}$  is  $O(n^{m+4})$ , where the constant in the  $O$ -notation is independent of  $n, m, k$ . On the other hand, the size of  $E_{n,k}$  is  $O(n^{m+1})$ . Thus the refutation size is polynomial in the size of the formula being refuted. ■

Let  $[n]^r$  [resp.  $[X]^r$ ] denote the set of  $r$ -element subsets of  $\{0, \dots, n-1\}$  [resp.  $X$ ]. The Erdős-Rado partition calculus notation  $n \rightarrow (m)_k^r$  means that for any partition  $f : [n]^r \rightarrow \{0, \dots, k-1\}$ , there is a size  $m$  subset  $X$  of  $\{0, \dots, n-1\}$  on which  $f$  is constant; i.e.  $f([X]^r)$  has cardinality 1. Using the Erdős-Rado partition calculus notation,  $PHP_n$  can be written as

$$n + 1 \rightarrow (2)_n^1$$

and Degen's principle as

$$mk + 1 \rightarrow (m + 1)_k^1.$$

**Question 7** *Are there polynomial size CP proofs of versions of Ramsey's theorem*

$$n + 1 \rightarrow (m)_k^r$$

*for appropriate  $n, m, r, k$ ?*

On p. 327 of [3], V. Chvatal gave a cutting planes proof of the instance

$$\begin{array}{l} 14 \not\rightarrow (3)_2^2 \\ 15 \rightarrow (3)_2^2 \end{array}$$

of Ramsey's theorem, and claimed that a general form of Ramsey's theorem could be proved in cutting planes. Since no details were given, it is unclear whether Chvatal's intended proof was indeed polynomial size. In [13], P. Pudlák has shown the existence of polynomial size constant depth Frege proofs of an appropriate formulation of Ramsey's theorem. It appears unlikely that  $CP$  (or  $CPLE$ , an extension of cutting planes with limited extension, see [2]) can polynomially simulate constant depth Frege systems. Hence it would be of interest to extend the counting arguments within  $CP$  to prove Ramsey's theorem and stronger combinatorial theorems.

## 4 Polynomial equivalence over resolution between Degen's principle and the pigeonhole principle

In the following we denote the cardinality of a finite set  $S$  by  $\|S\|$ .

**Definition 8** *The proof system  $Q$   $p$ -simulates the proof system  $P$  if there is a polynomial time computable function  $f$  such that for all  $x, y$ , if  $x$  is a  $P$ -proof of  $y$  then  $f(x, y)$  is a  $Q$ -proof of (the translation of)  $y$ . Proof systems  $P, Q$  are  $p$ -equivalent if each  $p$ -simulates the other.*

In this section, we show that over resolution, the clausal form of Degen's principle

$$(\forall f : (mk + 1) \rightarrow k)(\exists i < k)(\|f^{-1}(i)\| \geq m + 1)$$

is  $p$ -equivalent to the clausal form of the pigeonhole principle

$$(\forall g : (mk + 1) \rightarrow mk)(\exists i < mk)(\|g^{-1}(i)\| \geq 2).$$

(Note that we identify  $n$  with  $\{0, 1, \dots, n - 1\}$ ).

The idea of proof by contrapositives is quite simple. To show  $\neg D_{m,k} \Rightarrow \neg PHP_{mk}$ , suppose that  $f : (mk + 1) \rightarrow k$  violates Degen's principle; i.e.

$$(\forall j < k)(\|f^{-1}(j)\| \leq m) .$$

Define  $g : (mk + 1) \rightarrow mk$  as follows:  $g(i) = l \cdot k + j$  iff  $f(i) = j$  and this is the  $l + 1$ th such  $i$  (in case  $l = m - 1$  at least the  $m$ th such  $i$ ). In other words,  $g(i) = lk + j$  iff

$$\begin{array}{l} (l < m - 1 \wedge f(i) = j \wedge \|f^{-1}(j) \cap i\| = l) \text{ or} \\ (l = m - 1 \wedge f(i) = j \wedge \|f^{-1}(j) \cap i\| \geq l) . \end{array}$$

Since  $f$  violates Degen's principle, it follows that  $g$  is injective, so  $g$  violates the pigeonhole principle. To show  $\neg PHP_{mk} \Rightarrow \neg D_{m,k}$ , suppose that  $g : (mk+1) \rightarrow mk$  is injective. It follows that  $f(i) := g(i) \bmod k$  violates Degen's principle. In this section, the previous argument of equivalence is formalized within resolution.

Throughout this section, fix  $k, m$  and let  $n = mk$ . In the following definitions, the reader should bear in mind that in the sketch proof that  $\neg D_{m,k} \Rightarrow \neg PHP_{mk}$ , the  $p$ 's encode  $f$ , and the  $q$ 's encode  $g$ ; i.e.  $p_{i,j} = 1$  iff  $f(i) = j$  and  $q_{i,j} = 1$  iff  $g(i) = j$ .

**Definition 9** (a) In the following formulas denote the clauses equivalent to them. For instance  $\bigwedge_{i \in I} p'_i \rightarrow \bigvee_{i \in J} q'_i$  denotes the clause

$$\{\bar{p}'_i : i \in I\} \cup \{q'_i : i \in J\} .$$

(b) The set of defining clauses for  $p' \equiv \bigwedge_{i \in I} q'_i$  is the set of clauses

$$\{p' \rightarrow q'_i : i \in I\} \cup \{\bigwedge_{i \in I} q'_i \rightarrow p'\} .$$

The set of defining clauses for  $p' \equiv \bigvee_{i \in I} q'_i$  is the set

$$\{p' \rightarrow \bigvee_{i \in I} q'_i\} \cup \{q'_i \rightarrow p' : i \in I\} .$$

**Definition 10**  $\neg D_{m,k}$  is the set of clauses expressing the refutable version of Degen's principle

$$(\exists f : (n+1) \rightarrow k)(\forall i < k)(\|f^{-1}(i)\| \leq m)$$

so that

$$\neg D_{m,k} := D_{m,k}^{\neg,0} \cup D_{m,k}^{\neg,1}$$

with

$$D_{m,k}^{\neg,0} := \left\{ \bigvee_{j < k} p_{i,j} : i \leq n \right\}$$

and

$$D_{m,k}^{\neg,1} := \left\{ \bigvee_{i \in S} \bar{p}_{i,j} : j < k, S \in [n+1]^{m+1} \right\}.$$

**Definition 11** (a)  $[i]^{<m} := \bigcup_{l < m} [i]^l$ .

(b) If  $\|S\| < m - 1$ , then let  $\text{Def}(p_{i,j}^S)$  be the set of defining clauses for

$$p_{i,j}^S \equiv p_{i,j} \wedge \bigwedge_{i' \in S} p_{i',j} \wedge \bigwedge_{i' \in (i \setminus S)} \bar{p}_{i',j} .$$

If  $\|S\| = m - 1$ , let  $\text{Def}(p_{i,j}^S)$  be the set of defining clauses for

$$p_{i,j}^S \equiv p_{i,j} \wedge \bigwedge_{i' \in S} p_{i',j}.$$

(c)

$$\text{Def}(\vec{p}^S) := \bigcup_{i \leq n} \bigcup_{j < k} \bigcup_{S \in [i]^{< m}} \text{Def}(p_{i,j}^S).$$

(d) For  $i \leq n$ ,  $l < m$ ,  $j < k$  let  $\text{Def}(q_{i,j})$  be the set of defining clauses for

$$q_{i,lk+j} \equiv \bigvee_{S \in [i]^l} p_{i,j}^S$$

(e)

$$\text{Def}(\vec{q}) := \bigcup_{i \leq n} \bigcup_{j < n} \text{Def}(q_{i,j}) .$$

We have: If  $f(i) = j$  and  $f^{-1}(j) \cap i \supseteq S$ , then  $p_{i,j}^{S'}$  holds for some  $S' \supseteq S$  (which implies, that  $q_{i,lk+j}$  holds for some  $l$ ):

**Lemma 12** *If  $i \leq n$ ,  $j < k$ ,  $S \in [i]^{< m}$ , then*

$$D_{m,k}^{-,0}, \text{Def}(\vec{p}^S) \vdash (p_{i,j} \wedge \bigwedge_{i' \in S} p_{i',j}) \rightarrow \bigvee_{S \subseteq S' \in [i]^{< m}} p_{i,j}^S .$$

For all possible choices of  $i, j, S$  altogether we need  $\leq (n + 1)^{m+3}$  resolution steps.

**Proof** By induction on  $(m - 1) - \|S\|$ .

The base case  $\|S\| = m - 1$  is contained in  $\text{Def}(p_{i,j}^S)$ .

For the induction step, assume  $\|S\| = l < m - 1$ , and that the theorem is proven for  $\|S\| = l + 1$ . Let  $A := p_{i,j} \wedge \bigwedge_{i' \in S} p_{i',j}$ .

By  $\text{Def}(\vec{p}^S)$  we have

$$(A \wedge \bigwedge_{i' \in (i \setminus S)} \bar{p}_{i',j}) \rightarrow p_{i,j}^S .$$

By the induction hypothesis follows for  $i' \in (i \setminus S)$

$$(A \wedge p_{i',j}) \rightarrow \bigvee_{S \cup \{i'\} \subseteq S' \in [i]^{< m}} p_{i,j}^{S'} .$$

Now, applying  $i - \|S\|$  resolution steps we deduce the assertion.

To compute the number of resolution steps, note that for every  $i \leq n$ ,  $j < k$ ,  $l < m$ ,  $S \in [i]^l$ ,  $i - \|S\| \leq n$  steps are required, and  $\|[i]^l\| = \binom{i}{l} \leq (n + 1)^m$ ,

therefore altogether at most  $k \cdot (n+1) \cdot m \cdot (n+1)^m \cdot (n+1) = km(n+1)^{m+2} \leq (n+1)^{m+3}$  steps are required. ■

From lemma 12 follows now immediately, that  $\forall i \leq n \exists y < n. g(i) = y$  (namely  $g(i) = lk + j$  if  $p_{i,j}^S$  and  $\|S\| = l$ ):

**Lemma 13** For  $i \leq n$

$$\bigvee_{j < k} p_{i,j}, \text{Def}(\vec{p}^S), \text{Def}(\vec{q}) \vdash \bigvee_{y < n} q_{i,y}$$

for all  $i$  altogether in  $\leq 3(n+1)^{m+3}$  steps.

**Proof** By lemma 12, for each fixed  $i$  we have  $p_{i,j} \rightarrow \bigvee_{S \in [i] < m} p_{i,j}^S$ . From this and  $\bigvee_{j < k} p_{i,j}$  in  $k$  resolution steps we deduce

$$\bigvee_{S \in [i] < m} \bigvee_{j < k} p_{i,j}^S.$$

By  $\text{Def}(\vec{q})$   $p_{i,j}^S \rightarrow q_{i,k+\|S\|+j}$ . In  $\sum_{i \leq n} \sum_{j < k} \sum_{l < m} \|[i]^l\|$  resolution steps we conclude

$$\bigvee_{y < n} q_{i,y}.$$

Number of steps needed: For lemma 12 at most  $(n+1)^{m+3}$  steps, additionally at most  $k \cdot (n+1) \cdot (k + (n+1)^{m+1} \cdot m)$ , so altogether at most  $3(n+1)^{m+3}$  steps. ■

We show now that  $g$  is injective. First we have that we never get  $p_{i,j}^S$  and  $p_{i',j}^{S'}$  with  $i \neq i', \|S\| = \|S'\|$ :

**Lemma 14** Assume  $i < i' \leq n, j < k, S \in [i]^l, S' \in [i']^l$ . If  $l < m - 1$ , then

$$\text{Def}(\vec{p}^S) \vdash \bar{p}_{i,j}^S \vee \bar{p}_{i',j}^{S'}$$

and if  $l = m - 1$ , then

$$\text{Def}(\vec{p}^S), D_{m,k}^{-1,1} \vdash \bar{p}_{i,j}^S \vee \bar{p}_{i',j}^{S'}.$$

For all possible  $i, i', j, S$  together, at most  $(n+1)^{2m+3}(m+1)$  steps are needed.

**Proof** Case  $l < m - 1$ :

If  $S = S'$ , then, by  $\text{Def}(\vec{p}^S)$ , we have  $p_{i,j}^S \rightarrow p_{i,j}$ , since  $S' = S \subseteq i, i \notin S'$ , and therefore  $p_{i',j}^{S'} \rightarrow \bar{p}_{i',j}$ , by resolution  $\bar{p}_{i,j}^S \vee \bar{p}_{i',j}^{S'}$ .  
If  $S \neq S'$ , it follows, since  $\|S\| = \|S'\|$ , that  $S \not\subseteq S'$ , therefore there exists some

$i_0 \in (S \setminus S')$ , we have  $p_{i_0,j}^S \rightarrow p_{i_0,j}$ ,  $p_{i',j}^{S'} \rightarrow \bar{p}_{i_0,j}$ , again by resolution the assertion.

Case  $l = m - 1$ :

By  $D_{m,k}^{\neg,1}$

$$\bigvee_{i_0 \in S} \bar{p}_{i_0,j} \vee \bar{p}_{i,j} \vee \bar{p}_{i',j} ,$$

by  $\text{Def}(\vec{p}^S)$ , for  $i_0 \in S$ ,  $p_{i_0,j}^S \rightarrow p_{i_0,j}$ ,  $p_{i,j}^S \rightarrow p_{i,j}$ ,  $p_{i',j}^{S'} \rightarrow p_{i',j}$ , by  $m + 1$  resolution steps  $\bar{p}_{i,j}^S \vee \bar{p}_{i',j}^{S'}$ .

Number of steps needed: there are at most

$$\sum_{i < i' \leq n} \sum_{j < k} \sum_{l < m} \sum_{S \in [i]^l} \sum_{S' \in [i']^l} (m + 1) \leq km(m + 1)(n + 1)^{2m+2} \leq (n + 1)^{2m+3}(m + 1) \text{ steps.}$$

**Lemma 15** *If  $i < i' \leq n$ ,  $y < n$ ,*

$$\text{Def}(\vec{p}^S), \text{Def}(\vec{q}), D_{m,k}^{\neg,1} \vdash \bar{q}_{i,y} \vee \bar{q}_{i',y} .$$

*All proofs together need (if  $k \geq 2$ ) at most  $(n + 1)^{2m+4}$  steps.*

**Proof** By lemma 14, if  $l < m$ ,  $j < k$ ,  $S \in [i]^l$ ,  $S' \in [i']^l$  it follows that  $\bar{p}_{i,j}^S \vee \bar{p}_{i',j}^{S'}$ , further by  $\text{Def}(\vec{q})$  in  $\|[i]^{<m}\|$  resolution steps

$$q_{i,kl+j} \rightarrow \bigvee_{S \in [i]^l} p_{i,j}^S ,$$

therefore

$$q_{i,kl+j} \rightarrow \bar{p}_{i,j}^{S'}$$

by  $\text{Def}(\vec{q})$

$$q_{i',kl+j} \rightarrow \bigvee_{S' \in [i']^l} p_{i',j}^{S'} ,$$

by  $\|[i']^l\|$  steps  $\bar{q}_{i,kl+j} \vee \bar{q}_{i',kl+j}$ .

Number of Steps: steps from lemma 14 at most  $(m + 1)(n + 1)^{2m+3}$  steps, additionally

$$2 \sum_{i < i' \leq n} \sum_{l < m} \sum_{j < k} (\|[i]^{<m}\| \cdot \|[i']^{<m}\|) \leq 2(n + 1)^2 \cdot k \cdot m \cdot (n + 1)^{2m} \leq 2(n + 1)^{2m+3}$$

steps, altogether at most  $(n + 1)^{2m+3} \cdot (m + 3)$  steps. ■

This completes the formalization in resolution of  $\neg D_{m,k} \Rightarrow \neg PHP_{mk}$ . The formalization in resolution of the converse is straightforward and left to the reader. From the analysis of resolution steps and consideration of the number of variables which can appear in any clause, we deduce that  $D_{m,k} \equiv PHP_{mk}$  has resolution proofs of size polynomial in the size of  $D_{m,k}$  and  $PHP_{mk}$ .

Note that the proof of equivalence really uses extended resolution, since we introduced polynomially (in  $n$ ) many new literals  $p_{i,j}^S$  and  $q_{i,lk+j}$ . The  $p_{i,j}^S$  were defined in terms of the original  $p_{i,j}$ , and the  $q_{i,lk+j}$  were defined in terms of the  $p_{i,j}^S$ ; thus the depth of the extension is 2, and all definitions involve polynomially (in  $n$ ) many literals. By substituting the defining clauses appropriately, from a derivation of the empty clause from  $\neg PHP_{mk}$  we can obtain a derivation of the empty clause from  $\neg D_{m,k}$ . It is in this sense we mean that over resolution  $PHP_{mk}$  and  $D_{m,k}$  are polynomially equivalent.

Since cutting planes  $p$ -simulates resolution, it follows that  $D_{m,k} \equiv PHP_{mk}$  has polynomial size cutting planes proofs (more precisely, cutting planes with extension, where the depth of the extension is 2, and all extending inequalities involve polynomially (in  $n$ ) many literals; see [5] for information on cutting planes with extension). Since  $PHP_{mk}$  is well-known to have cutting planes proofs of size polynomial in  $mk$ , we have an alternative proof of the main result of the previous section.

## 5 $st$ -Connectivity

Graph connectivity has been studied under various guises by many authors (Floyd-Warshall's  $O(n^2 \log n)$  algorithm for transitive closure, the well-known observation that the so-called directed *graph accessibility problem* GAP is complete for nondeterministic logarithmic space, Borodin's observation that LOGSPACE is contained in  $AC^1$ , etc.). In [10], M. Karchmer and A. Wigderson gave a significant size lower bound for boolean circuits computing *st-connectivity*, the problem whether there is a path from designated nodes  $s$  to  $t$  in an undirected graph.

**Theorem 16 (Karchmer-Wigderson [10])** *Monotonic fan-in 2 depth of st-connectivity is  $\Theta(\log^2 n)$ .*

In [18], I. Wegener proved the monotonic analogue of Spira's theorem which relates depth and size of monotonic formulas: a problem  $P$  has depth  $O(d)$  monotonic formulas if and only if  $P$  has size  $2^{O(d)}$  monotonic circuits (see Boppana-Sipser [1] for an overview of boolean circuit complexity). Wegener's result, with the previous theorem, implies the following.

**Corollary 17 (Karchmer-Wigderson [10])** *Monotonic formula size of st-connectivity is  $\Theta(n^{\log n})$ .*

Another application of the previous theorem, not used in this paper, is the following result.

**Theorem 18 (Clote [4])** *The monotonic depth for fan-in 2 circuits recognizing whether a 2-CNF formula is refutable is  $\Theta(\log^2 n)$ .*

There are various possible formulations of  $st$ -connectivity in propositional logic.

**Definition 19 ( $st$ -connectivity (Form 1))** *Assume that  $G$  is a finite undirected graph, with two designated vertices  $s, t$  of degree 1, while all other vertices have degree 2. Then there is a path from  $s$  to  $t$ .*

This notion of  $st$ -connectivity, proposed by P. Pudlák, was investigated by S. Buss and P. Clote in [2] (this version of  $st$ -connectivity for *directed graphs* was there shown to be equivalent (over constant depth, polynomial size Frege systems) to the onto version of the pigeonhole principle, and polynomial size  $CP$  proofs were given for  $st$ -connectivity for undirected graphs). An alternate, weaker form of  $st$ -connectivity is now defined.

**Definition 20 ( $st$ -connectivity (Form 2))** *Assume that  $G$  is a finite undirected graph with two distinct, designated vertices  $s, t$ . Then either there is a path from  $s$  to  $t$ , or there is a partition of the vertices of  $G$  into two classes, where  $s$  and  $t$  lie in different classes and no edge goes between vertices lying in different classes.*

We will show that there are polynomial size resolution proofs for  $st$ -connectivity (Form 2), though of course, since  $st$ -connectivity (Form 1) implies the pigeonhole principle, there is an exponential lower bound for resolution proof size for  $st$ -connectivity (Form 1). The formalization in resolution of Definition 20 is now given. The formula  $A(\vec{p}, \vec{q})$  is the conjunction of the following clauses:

1.  $q_{0,0}$
2.  $q_{n+1,n+1}$
3.  $\bar{q}_{i,j}, \bar{q}_{i,k}$ , for all  $j \neq k$  in  $\{0, \dots, n+1\}$ .
4.  $q_{i,0}, \dots, q_{i,n+1}$ , for all  $i \in \{1, \dots, n\}$ .
5.  $\bar{q}_{i,j}, \bar{q}_{i+1,k}, p_{j,k}$ , for all  $j \neq k$  in  $\{0, \dots, n+1\}$ .
6.  $\bar{p}_{i,j}, p_{j,i}$ , for all  $i \neq j$  in  $\{0, \dots, n+1\}$ .

The idea is that the  $p$ 's express the edge relation of  $G$  ( $p_{i,j} = 1$  iff there is a directed edge from  $i$  to  $j$ ), and that the  $q$ 's define a path from  $s = 0$  to  $t = n+1$ . [We allow multiple occurrences of the same vertex along a path.] Thus  $A(\vec{p}, \vec{q})$  expresses that  $G$  is a directed graph and there is a path from  $s$  to  $t$ .

The formula  $B(\vec{p}, \vec{r})$  is the conjunction of the following clauses:

1.  $\bar{r}_0$
2.  $r_{n+1}$
3.  $\bar{r}_i, \bar{p}_{i,j}, r_j$ , for all  $i \neq j$  in  $\{0, \dots, n+1\}$ .

The idea is that the  $p$ 's express the edge relation of  $G$ , and the  $r$ 's express the partition: those vertices  $i$  in the same partition class as  $s$  (we identify  $s$  with 0) satisfy  $\bar{r}_i$ , while those in the same class as  $t$  (we identify  $t$  with  $n + 1$ ) satisfy  $r_i$ . Thus  $B(\vec{p}, \vec{r})$  expresses that there is a *cut* (or partition into 2 classes), for which  $s$  [resp.  $t$ ] belongs to [resp. does not belong to] the cut, and such that no edge goes from a vertex belonging to the cut to a vertex not belonging to the cut.

The resolution formulation of  $st$ -connectivity (Form 2) is the conjunction of both  $A(\vec{p}, \vec{q})$ , which expresses that either  $G$  is not an undirected graph, or there is a path from  $s$  to  $t$ , and  $B(\vec{p}, \vec{r})$ , which states that there is a partition of  $G$ 's vertices, with  $s, t$  in different classes, and for which no edge of  $G$  goes between vertices in different classes. Note that all occurrences of  $\vec{p}$  in the clauses  $B$  are negative.

J. Krajíček [12] proved an interesting interpolation result, which relates monotonic circuit lower bounds with lower bounds resolution proofs (P. Pudlák [14] extended this to an interpolation result relating monotonic real circuit lower bounds with lower bounds for cutting plane proofs).

**Theorem 21 (Krajíček [12])** *Suppose that propositional variables  $\vec{p}$  are positive in  $A(\vec{p}, \vec{q})$ , or that  $\vec{p}$  are negative in  $B(\vec{p}, \vec{r})$ , and that there is a resolution refutation  $P$  of  $A(\vec{p}, \vec{q}) \wedge B(\vec{p}, \vec{r})$  of depth  $d$  and size  $s$ . Then there is a monotonic boolean circuit  $C$  of depth  $d$  and size  $O(s)$  for which*

$$C(\vec{p}) = \begin{cases} 0 & \text{if } A(\vec{p}, \vec{q}) \text{ is refutable} \\ 1 & \text{else if } B(\vec{p}, \vec{r}) \text{ is refutable} \end{cases}$$

Moreover, if  $P$  is a proof tree, then the circuit  $C$  has fan-out 1.

By Theorem 21 and Corollary 17 we have the following.

**Theorem 22** *All tree-like resolution proofs of  $st$ -connectivity (Form 2) have size  $\Omega(n^{\log n})$ .*

**Proof** Let  $s(n)$  be the size of a refutation of  $A(\vec{p}, \vec{q}) \wedge B(\vec{p}, \vec{r})$ , where the latter expresses the statement that a directed graph  $G$  on  $n$  nodes has a path from  $s$  to  $t$  and at the same time a cut between  $s$  and  $t$ . From Theorem 21 there are monotone boolean circuits  $C$  of size  $O(s(n))$  such that  $C(\vec{a}) = 0$  implies that  $A(\vec{a}, \vec{q})$  is refutable and  $C(\vec{a}) = 1$  implies that  $B(\vec{a}, \vec{r})$  is refutable. Restricting  $C$  to those inputs  $\vec{a}$  which correctly encode a directed graph  $G$ , it follows that  $C(\vec{a}) = 1$  if and only if there is a path from  $s$  to  $t$ . By Corollary 17, it follows that  $s(n)$  must be  $\Omega(n^{\log n})$ . ■

The separation between tree-like resolution and resolution is a corollary of the following.

**Theorem 23** *There are polynomial size resolution proofs of  $st$ -connectivity (Form 2).*

**Proof** We begin by the following claim.

**Claim:** For  $1 \leq i \leq n+1$ , there is a resolution proof of  $\bar{q}_{i,j}, \bar{r}_j$

The proof of the claim is by induction on  $i$ . For the base case of  $i = 1$ , note that

$$\frac{\frac{\bar{q}_{0,0}, \bar{q}_{1,k}, p_{0,k}}{\bar{q}_{0,0}, \bar{q}_{1,k}, \bar{r}_k, r_0} \quad \frac{\bar{r}_k, \bar{p}_{k,0}, r_0 \quad \bar{p}_{0,k}, p_{k,0}}{\bar{r}_k, \bar{p}_{0,k}, r_0}}{\bar{q}_{1,k}, \bar{r}_k, r_0} \quad q_{0,0}}{\bar{q}_{1,k}, \bar{r}_k} \quad \bar{r}_0$$

The resolution proof for the base case is  $O(n)$  size. Now, the induction hypothesis is

$$\bar{q}_{i,j}, \bar{r}_j.$$

We have the following auxiliary result.

$$\frac{\frac{\bar{q}_{i,j}, \bar{q}_{i+1,k}, p_{j,k}}{\bar{q}_{i,j}, \bar{q}_{i+1,k}, \bar{r}_k, r_j} \quad \frac{\bar{p}_{j,k}, p_{k,j} \quad \bar{r}_k, \bar{p}_{k,j}, r_j}{\bar{r}_k, \bar{p}_{j,k}, r_j}}{\bar{q}_{i,j}, \bar{q}_{i+1,k}, \bar{r}_k}$$

Now

$$\frac{\frac{q_{i,0}, q_{i,1}, \dots, q_{i,n+1} \quad \bar{q}_{i,0}, \bar{q}_{i+1,k}, \bar{r}_k}{q_{i,1}, q_{i,2}, \dots, q_{i,n+1}, \bar{q}_{i+1,k}, \bar{r}_k} \quad \bar{q}_{i,1}, \bar{q}_{i+1,k}, \bar{r}_k}{q_{i,2}, \dots, q_{i,n+1}, \bar{q}_{i+1,k}, \bar{r}_k}$$

Inductively continuing in this manner, we obtain

$$\bar{q}_{i+1,k}, \bar{r}_k.$$

This completes the inductive case. For  $i, k$  fixed, there are  $O(n)$  additional resolution steps, with overall size  $O(n^2)$ .

Taking  $i = n+1$ , it follows that

$$\bar{q}_{n+1,k}, \bar{r}_k$$

for all  $k$ , so that

$$\frac{\frac{\bar{q}_{n+1,n+1}, \bar{r}_{n+1} \quad q_{n+1,n+1}}{\bar{r}_{n+1}} \quad r_{n+1}}{\emptyset}$$

We have thus derived the empty clause by a proof of size  $O(n^4)$  from the assumptions. ■

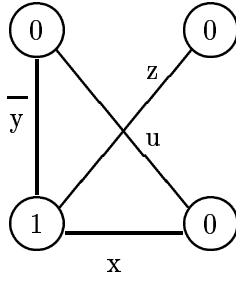


Figure 1: Odd-charged graph with edges labeled by literals

**Corollary 24** *Tree-like resolution does not polynomially simulate resolution.*

See A. Urquhart’s survey article [17] for discussion of Tseitin’s original argument.

## 6 Tseitin’s odd-charged graphs

In [15], Tseitin associated propositional formulas with labeled undirected graphs, and developed a technique for obtaining lower bounds for regular resolution refutations (regular resolution allows, on any branch of the refutation tree, at most one resolution on any particular variable).

Suppose that  $G$  is a finite, undirected graph, whose vertices are labeled by 0, 1 and whose edges are labeled by distinct propositional literals (if literal  $\alpha$  labels edge  $e$ , then neither  $\alpha$  nor  $\bar{\alpha}$  can label another edge). The label on a vertex is said to be its charge. The graph  $G$  is said to be odd-charged, if the sum modulo 2 of the vertex labels is 1. Figure 1 is an example.

Associate with  $G$  its charge equations, i.e. for vertex  $v$  the equation  $EQ(v)$  states that the sum modulo 2 of the literals incident to  $v$  equals the charge on  $v$ . For the example in Figure 1, here are the charge equations:

1.  $\bar{y} \oplus u = 0$
2.  $\bar{y} \oplus x \oplus z = 1$
3.  $z = 0$
4.  $x \oplus u = 0$

The Tseitin clauses  $F(G)$  associated with graph  $G$  are the clauses corresponding to the  $CNF$  formulation of the charge equations. With this example, we have:

1.  $\{\bar{u}, \bar{y}\}, \{u, y\}$
2.  $\{x, y, \bar{z}\}, \{x, \bar{y}, z\}, \{\bar{x}, y, z\}, \{\bar{x}, \bar{y}, \bar{z}\}$

3.  $\{\bar{z}\}$
4.  $\{x, \bar{u}\}, \{\bar{x}, u\}$

The rule for producing clauses from a charge equation is to place an odd [resp. even] number of negations on the associated literals, if the charge is 0 [resp. 1]. Clearly, there are  $2^{d-1}$  clauses associated with the charge equation for vertex  $v$  if the degree of  $v$  is  $d$  (note that half of the  $2^d$  truth assignments satisfy the charge equation). When considering proof size, we are thus only interested in graph families of bounded degree. The key property of odd-charged graphs is given by the following.

**Fact 25 (Tseitin [15, 17])** *The connected graph  $G$  is odd-charged if and only if the clauses  $F(G)$  are unsatisfiable.*

In [15], Tseitin developed recurrence relations for regular resolution refutation size, depending on a connectivity parameter for the graphs, and thus proved an exponential lower bound for regular resolution refutations for  $F(L_n)$ , where  $L_n$  is the odd-charged  $n \times n$  lattice. In [16], A. Urquhart employed A. Haken's lower bound technique [8] to prove an exponential lower bound for Tseitin tautologies associated with particular expander graphs. Two interesting questions remain in this area:

1. Are there polynomial size  $CP$  refutations for Urquhart's formulas [16]?
2. For which families of graphs of bounded degree are there superpolynomial lower bounds for resolution [resp. regular resolution] refutations of Tseitin formulas?

Despite Tseitin's recurrence relations, it seems to be an interesting open problem to determine which graph theoretic properties lead to polynomial size regular resolution proofs.

**Lemma 26** *Let  $u, v$  be two nodes of a charged, labeled, undirected graph  $G$ , which are joined by an edge in  $G$  labeled by  $x$ . Let  $G'$  be obtained from  $G$  by contracting the edge  $\{u, v\}$ . In other words, define  $V(G') = V(G) - \{u, v\} \cup \{w\}$  where  $w \notin V(G)$  is a new node, and*

$$E(G') = E(G) - \{e : u \in e \text{ or } v \in e\} \cup \{\{r, w\} : \{r, u\} \in E(G) \text{ or } \{r, v\} \in E(G)\}.$$

*The charge on every node in  $V(G') - \{w\}$  is the same as the charge of that node in  $G$ , while the charge on  $w$  is defined to be*

$$\text{charge}(u) \oplus \text{charge}(v).$$

*Then there are  $2^{dg(u)+dg(v)-3}$  resolution steps to derive the clauses associated with the charge equation  $EQ(w)$  in  $G'$  from the charge equations  $EQ(u)$ ,  $EQ(v)$  in  $G$ .*

The proof of this lemma follows from a simple computation, where positive [resp. negative] occurrences of the edge literal  $x$  in  $EQ(u)$  are resolved against negative [resp. positive] occurrences of  $x$  in  $EQ(v)$ . The formal proof is left to the reader.

**Definition 27** *The graph  $H$  is a minor of the graph  $G$ , if  $H$  can be obtained from  $G$  by the operations of deleting an isolated vertex, removing an edge, contracting an edge.*

Edge deletions and their relation to regular resolution were first discussed by Tseitin [15, 17], and were the basis of his recurrence relations. Isolated vertices play no role in Tseitin's formulas, as there is no charge equation for such vertices. Finally, edge contractions can be handled by the previous lemma. From this discussion, it is clear that a precise relation between  $F(G)$  and  $F(H)$  can be worked out, if  $H$  is a minor of  $G$ . Moreover, a simple computation, contracting edges beginning with those adjacent to the leaves, shows that the Tseitin formulas  $F(G_n)$  have linear size regular resolution refutations, for families  $\{G_n : n \in \mathbf{N}\}$  of bounded degree odd-charged trees. This follows from the next proposition.

**Proposition 28** *Let  $T$  be a rooted, odd-charged binary tree with  $n$  nodes and degree bound  $e$ . Then there are regular resolution refutations of  $F(T)$  consisting of at most  $n(2^e - 1)$  steps.*

**Proof** Contract edges, beginning with the leaves of  $T$ . By Lemma 26, the number of resolution steps required to contract the edge connecting a leaf (of degree 1) with its parent is  $2^{e-1}$ . For that parent, there are respectively  $2^{e-2}$ ,  $2^{e-3}$ , etc. many steps in contracting sibling leaves with the same parent. Thus an upper bound for the number of resolution steps required to derive the empty clause is the number of internal nodes of  $T$  times  $\sum_{i=0}^{e-1} 2^i$ , so bounded by  $n(2^e - 1)$ . ■

Hence, it seems possible that one could classify via graph theoretic properties those families of bounded degree connected graphs whose associated Tseitin formulas have polynomial size regular resolution refutations. Noting the correspondence in regular resolution with operations in the definition of graph minor, certain techniques may be applicable from the Robertson-Seymour theorem that the collection of all finite graphs is well quasi-ordered under graph minors.

As a small step in this direction, we describe degree 3 graphs whose Tseitin formulas correspond to the pigeonhole principle, and for which there are no polynomial size resolution (or even constant depth Frege) proofs. As shown in Figure 2, the graph  $G_n$  consists of  $n + 1$  top trees  $B_i$ , and  $n$  bottom trees  $C_j$ , both indicated by triangles.  $B_i$  is responsible for mapping the  $i$ -th pigeon, and  $C_j$  is responsible for the  $j$ -th hole. The  $j$ th leaf of  $B_i$  is connected with the  $i$ th

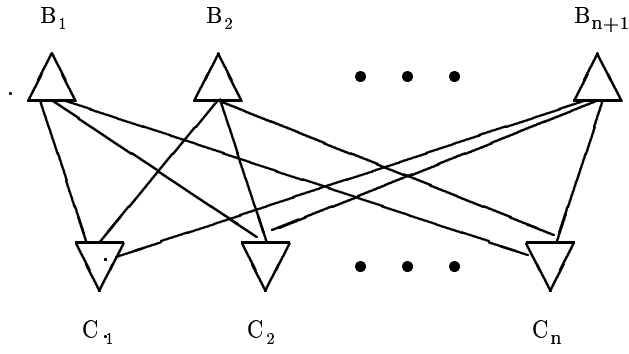


Figure 2: Global view of Tseitin graph representing  $PHP_n$

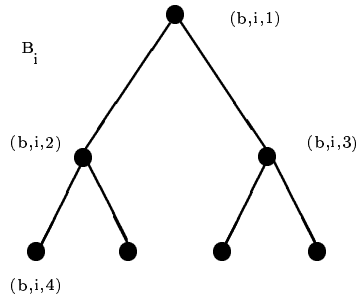


Figure 3: Local view of Tseitin graphs for the  $B$ 's

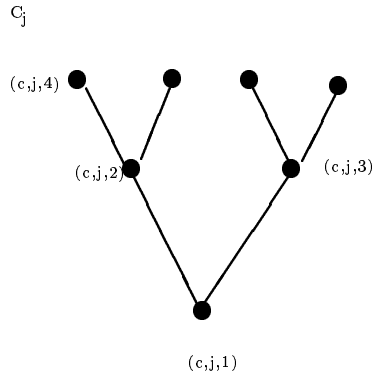


Figure 4: Local view of Tseitin graphs for  $C$ 's

leaf of  $C_j$ . The  $B$ 's themselves are shown in Figure 3 and the  $C$ 's are shown in Figure 4. The trees  $B_i$  have  $n$  leaves each, therefore  $2n - 1$  nodes, whereas the trees  $C_i$  have  $n + 1$  leaves and  $2n + 1$  nodes. The  $j$ th leaf of  $B_i$  has number  $\text{leaf}_B(j) := n - 1 + j$  and the  $i$ th leaf of  $C_j$  has number  $\text{leaf}_C(i) := n + i$ . Therefore  $G_n$  has vertices

$$\{\langle b, i, j \rangle : 1 \leq i \leq n + 1, 1 \leq j \leq 2n - 1\} \cup \{\langle c, i, j \rangle : 1 \leq i \leq n, 1 \leq j \leq 2n + 1\}$$

and edges from the following three sets:

$$\{\{\langle b, i, \lfloor j/2 \rfloor \rangle, \langle b, i, j \rangle\} : 1 \leq i \leq n + 1, 2 \leq j \leq 2n - 1\}$$

and

$$\{\{\langle c, i, \lfloor j/2 \rfloor \rangle, \langle c, i, j \rangle\} : 1 \leq i \leq n, 2 \leq j \leq 2n + 1\}$$

and

$$\{\{\langle b, i, \text{leaf}_B(j) \rangle, \langle c, j, \text{leaf}_C(i) \rangle\} : 1 \leq i \leq n + 1, 1 \leq j \leq n\}.$$

We assign propositional variables to the edges of  $G_n$  as follows:  $p_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle}$  labels the edge from  $\lfloor j/2 \rfloor$  to  $j$  in the tree  $B_i$ ;  $q_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle}$  labels the edge from  $\lfloor j/2 \rfloor$  to  $j$  in the tree  $C_i$ ; and  $r_{\langle i, \text{leaf}_B(j) \rangle, \langle j, \text{leaf}_C(i) \rangle}$  is responsible for connecting leaf  $j$  of  $B_i$  to leaf  $i$  of  $C_j$ . Here, the propositional variable sets are given as follows:

$$\{p_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle} : 1 \leq i \leq n + 1, 2 \leq j \leq 2n - 1\}$$

and

$$\{q_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle} : 1 \leq i \leq n, 2 \leq j \leq 2n + 1\}$$

and

$$\{r_{\langle i, \text{leaf}_C(j) \rangle, \langle j, \text{leaf}_B(i) \rangle} : 1 \leq i \leq n + 1, 1 \leq j \leq n\}.$$

Let each of  $\langle b, i, 1 \rangle$  for  $1 \leq i \leq n + 1$ , and  $\langle c, j, 1 \rangle$  for  $1 \leq j \leq n$  have charge 1, and all other nodes have charge 0. The edges of  $G_n$  are labeled in the obvious manner (eg.  $\{\langle b, i, \lfloor j/2 \rfloor \rangle, \langle b, i, j \rangle\}$  is labeled by  $p_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle}$  etc.). Therefore  $G$  has odd charge  $2n + 1$  and we have the following charge equations:

$$(1) \quad p_{\langle i, 1 \rangle, \langle i, 2 \rangle} \oplus p_{\langle i, 1 \rangle, \langle i, 3 \rangle} = 1$$

for all  $1 \leq i \leq n + 1$ ,

$$(2) \quad q_{\langle i, 1 \rangle, \langle i, 2 \rangle} \oplus q_{\langle i, 1 \rangle, \langle i, 3 \rangle} = 1$$

for all  $1 \leq i \leq n$ ,

$$(3) \quad p_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle} \oplus p_{\langle i, j \rangle, \langle i, 2j \rangle} \oplus p_{\langle i, j \rangle, \langle i, 2j+1 \rangle} = 0$$

for all  $1 \leq i \leq n+1, 2 \leq j \leq n-1$ ,

$$(4) \quad q_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle} \oplus q_{\langle i, j \rangle, \langle i, 2j \rangle} \oplus q_{\langle i, j \rangle, \langle i, 2j+1 \rangle} = 0$$

for all  $1 \leq i \leq n, 2 \leq j \leq n$ ,

$$(5) \quad p_{\langle i, \lfloor \frac{\text{leaf}_B(j)}{2} \rfloor \rangle, \langle i, \text{leaf}_B(j) \rangle} \oplus r_{\langle i, \text{leaf}_B(j) \rangle, \langle j, \text{leaf}_C(i) \rangle} = 0$$

for all  $1 \leq i \leq n+1$ , and  $1 \leq j \leq n$ ,

$$(6) \quad r_{\langle i, \text{leaf}_B(j) \rangle, \langle j, \text{leaf}_C(i) \rangle} \oplus q_{\langle j, \lfloor \frac{\text{leaf}_C(i)}{2} \rfloor \rangle, \langle j, \text{leaf}_C(i) \rangle} = 0$$

for all  $1 \leq i \leq n+1$ , and  $1 \leq j \leq n$ .

The set of  $G_n$ 's vertices has cardinality  $(n+1)(2n-1) + n(2n+1) = 0(n^2)$  and each node has degree at most 3.

We claim that there are polynomial size resolution derivations of  $F(G_n)$  from  $\neg PHP_n$  – recall that the Tseitin formula  $F(G_n)$  is refutable, whereas  $PHP_n$  is a tautology.

To this end, for  $1 \leq i \leq n+1, 2 \leq j \leq 2n-1$  let

$$(7) \quad p_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle} \equiv \bigvee_{\{1 \leq k \leq n: j \sqsubseteq \text{leaf}_B(k)\}} P_{i,k}$$

where  $u \sqsubseteq v$  means that  $u$  is a prefix of  $v$  (integer  $u$  is a prefix of integer  $v$  if the binary representation of  $u$  is a prefix of the binary representation of  $v$ ). The idea is that the leaves of the tree  $B_{i_0}$  are labeled by  $P_{i_0,1}, P_{i_0,2}, \dots, P_{i_0,n}$  and that an edge  $p_{\langle i, \lfloor j/2 \rfloor \rangle, \langle i, j \rangle}$  between node  $\langle b, i, \lfloor j/2 \rfloor \rangle$  and node  $\langle b, i, j \rangle$  of  $B_{i_0}$  has the value 1 iff  $\langle b, i, \lfloor j/2 \rfloor \rangle$  is the ancestor of a leaf bearing the value 1. For  $1 \leq j \leq n, 2 \leq i \leq 2n+1$  let

$$(8) \quad q_{\langle j, \lfloor i/2 \rfloor \rangle, \langle j, i \rangle} \equiv \bigvee_{\{1 \leq k \leq n+1: i \sqsubseteq \text{leaf}_C(k)\}} P_{k,j}$$

Similarly, the idea is that the leaves of the tree  $C_{j_0}$  are labeled by  $P_{1,j_0}, P_{2,j_0}, \dots, P_{n+1,j_0}$  and that an edge  $q_{\langle j, \lfloor i/2 \rfloor \rangle, \langle j, i \rangle}$  between internal nodes of  $C_{j_0}$  has the value 1 iff  $\langle c, j, \lfloor i/2 \rfloor \rangle$  is the ancestor of a leaf bearing the value 1. For  $1 \leq i \leq n+1, 1 \leq j \leq n$  let

$$(9) \quad r_{\langle i, \text{leaf}_B(j) \rangle, \langle j, \text{leaf}_C(i) \rangle} \equiv P_{i,j}.$$

The function of the  $r$ 's is to connect up leaves of the  $B$ 's with those of the  $C$ 's, where leaf the  $j$ -th leaf of  $B_i$  (labeled by  $P_{i,j}$ ) is connected to the  $i$ -th leaf of  $C_j$ . Finally, let  $DEF(G_n)$  be the resolution clauses corresponding (as in Section 4) to the above definitions.

**Definition 29** *The negation of the onto-version of the pigeonhole principle, denoted  $\neg PHP_n^{onto}$ , states that there is a bijection from  $\{1, \dots, n+1\}$  onto  $\{1, \dots, n\}$ , and is given by the following clauses:*

$$\{\{P_{i,1}, \dots, P_{i,n}\} : 1 \leq i \leq n+1\} \cup \{\{\bar{P}_{i,j}, \bar{P}_{i',j}\} : 1 \leq i < i' \leq n+1, 1 \leq j \leq n\}$$

together with

$$\{\{P_{1,j}, \dots, P_{n+1,j}\} : 1 \leq j \leq n\} \cup \{\{\bar{P}_{i,j}, \bar{P}_{i,j'}\} : 1 \leq i \leq n+1, 1 \leq j < j' \leq n\}.$$

Note that the onto version of the pigeonhole principle requires a bijection (not simply injection) from the domain of size  $n+1$  to range of size  $n$ . In contrast to the formalization of  $PHP_n$ , we have  $1 \leq i \leq n+1$  and  $1 \leq j \leq n$  to allow a simple correspondence with the trees  $B_i$  and  $C_j$ .

**Theorem 30** *There are polynomial size resolution derivations of  $F(G_n)$  from  $DEF(G_n)$  and  $\neg PHP_n^{onto}$ .*

**Proof** For  $1 \leq j \leq n$ , by Equation (7)

$$P_{\langle i, \lfloor \frac{\text{leaf}_B(j)}{2} \rfloor \rangle, \langle i, \text{leaf}_B(j) \rangle} \equiv P_{i,j}$$

and by Equation (9)

$$r_{\langle i, \text{leaf}_B(j) \rangle, \langle j, \text{leaf}_C(i) \rangle} \equiv P_{\langle i, \lfloor \frac{\text{leaf}_B(j)}{2} \rfloor \rangle, \langle i, \text{leaf}_B(j) \rangle}.$$

This proves Equation (5). Equation (6) is similarly derived. Thus we've established the charge equations for the connection between appropriate trees  $B_i$  to  $C_j$ .

Fix  $1 \leq i \leq n+1$ . Equations (1) and (3) respectively correspond to the charge equation at the root and the charge equations at non-root internal nodes of the tree  $B_i$ . Consider first Equation (1). To simplify notation, write  $d$  resp.  $e$  in place of  $p_{\langle i,1 \rangle, \langle i,2 \rangle}$  resp.  $p_{\langle i,1 \rangle, \langle i,3 \rangle}$ , and let  $P_a, P_{a+1}, \dots, P_b$  resp.  $P_{b+1}, \dots, P_c$  denote the leaf labels in  $B_i$  below  $d$  resp.  $e$  (hence the  $P$ 's correspond to appropriate  $P_{i,j}$ 's). With this notation, Equation (7) states that  $d \equiv P_a \vee \dots \vee P_b$  and  $e \equiv P_{b+1} \vee \dots \vee P_c$ , so from its clausal form in  $DEF(G_n)$ ,

$$\{\bar{d}, P_a, \dots, P_b\}.$$

Repeatedly resolve this clause against clauses

$$\{\bar{P}_a, \bar{P}_{b+1}\}, \{\bar{P}_{a+1}, \bar{P}_{b+1}\}, \{\bar{P}_{a+2}, \bar{P}_{b+1}\}, \dots, \{\bar{P}_b, \bar{P}_{b+1}\}$$

(these clauses come from  $\neg PHP_n^{onto}$ ) to obtain  $\{\bar{d}, \bar{P}_{b+1}\}$ . In a similar fashion, obtain  $\{\bar{d}, \bar{P}_{b+2}\}, \{\bar{d}, \bar{P}_{b+3}\}, \dots, \{\bar{d}, \bar{P}_c\}$ . From  $DEF(G_n)$ , we have  $\{\bar{e}, P_{b+1}, \dots, P_c\}$ , so by repeated resolution against the previous clauses, we obtain  $\{\bar{d}, \bar{e}\}$ .

From  $\neg PHP_n^{onto}$ , we have  $\{P_a, \dots, P_c\}$ , and from  $DEF(G_n)$  we have  $\{\bar{P}_a, d\}, \dots, \{\bar{P}_b, d\}$  and  $\{\bar{P}_{b+1}, e\}, \dots, \{\bar{P}_c, e\}$ . Repeatedly resolving the former against the latter, we obtain  $\{d, e\}$ . Now  $\{d, e\}$  and  $\{\bar{d}, \bar{e}\}$  form the clausal representation of the charge equation (1)  $d \oplus e = 1$ . In a similar fashion (using the onto part of the formulation of the pigeonhole principle) one can prove Equation (2). This concludes the derivation of charge equations for roots of the  $B_i$  and  $C_j$ .

Consider now Equation (3). To simplify notation, write  $d, e, f$  resp. for  $\mathcal{P}^{(i, \lfloor j/2 \rfloor), (i, j)}, \mathcal{P}^{(i, j), (i, 2j)}, \mathcal{P}^{(i, j), (i, 2j+1)}$ , and let  $P_a, \dots, P_b$  resp.  $P_{b+1}, \dots, P_c$  denote the leaves of tree  $B_i$  respectively below  $e, f$ . Thus the leaves below  $d$  are  $P_a, \dots, P_c$ . The clausal representation of Equation (3) has the following clauses:

1.  $\{\bar{d}, e, f\}$
2.  $\{d, \bar{e}, f\}$
3.  $\{d, e, \bar{f}\}$
4.  $\{\bar{d}, \bar{e}, \bar{f}\}$ .

The first clause is simple to obtain. From  $DEF(G_n)$  we have  $\{\bar{d}, P_a, \dots, P_c\}$  and  $\{\bar{P}_a, e\}, \{\bar{P}_{a+1}, e\}, \dots, \{\bar{P}_b, e\}$  and  $\{\bar{P}_{b+1}, f\}, \{\bar{P}_{b+2}, f\}, \dots, \{\bar{P}_c, f\}$ . By resolution of the former against the latter, we have  $\{\bar{d}, e, f\}$ . The second and third clauses are similarly derived. To obtain the fourth clause, proceed as follows. By  $DEF(G_n)$ , we have  $\{\bar{e}, P_a, \dots, P_b\}$  and from  $\neg PHP_n^{onto}$  we have  $\{\bar{P}_a, \bar{P}_{b+1}\}, \{\bar{P}_{a+1}, \bar{P}_{b+1}\}, \dots, \{\bar{P}_b, \bar{P}_{b+1}\}$  and so by resolution we obtain  $\{\bar{e}, \bar{P}_{b+1}\}$ . Similarly, we obtain  $\{\bar{e}, \bar{P}_{b+2}\}, \dots, \{\bar{e}, \bar{P}_c\}$ , and in an analogous fashion  $\{\bar{f}, \bar{P}_a\}, \dots, \{\bar{f}, \bar{P}_b\}$ . By  $DEF(G_n)$ , we have  $\{\bar{d}, P_a, \dots, P_c\}$  and so by repeated resolution against the preceding clauses, we derive  $\{\bar{d}, \bar{e}, \bar{f}\}$ , as required. This establishes Equation (3). The derivation of Equation (4) is analogous. This completes our treatment of the charge equations for non-root internal nodes of trees  $B_i$  and  $C_j$ .

Thus we have a resolution derivation of  $F(G_n)$  from  $DEF(G_n)$  and  $\neg PHP_n^{onto}$ . Straightforward estimation shows that the sketched resolution proof is of polynomial size. ■

**Corollary 31** *There is an exponential lower bound for resolution (even constant depth Frege) refutations of  $F(G_n)$ .*

The corollary is immediate, since it is well-known that  $\neg PHP_n^{onto}$  has an exponential size lower bound for resolution (and constant depth Frege) refutations. See [11] for details.

From this construction, one might think that for every unsatisfiable propositional formula  $H$  there is a related odd charged graph  $G$ , for which  $H \rightarrow F(G)$  has a polynomial size resolution (or constant depth Frege) derivation. This however is false, unless  $NP = co - NP$ .

**Proposition 32** *For any polynomials  $p, q$  there exists an unsatisfiable propositional formula  $H$  such that for all odd charged graphs  $G$  of size at most  $q(|H|)$ , there is a resolution (or constant depth Frege, or Frege, etc.) derivation of  $H \rightarrow F(G)$ , where  $F(G)$  is the Tseitin formula related to  $G$ .*

**Proof** If not, then we have an  $NP$ -procedure to test whether a formula  $\phi$  is a tautology:  $\phi \in \text{TAUT}$  iff  $\neg\phi \notin \text{SAT}$  iff there is odd charged  $G$  of size  $q(|\phi|)$  and a resolution (or constant depth Frege, or Frege, etc.) derivation of  $\neg\phi \rightarrow F(G)$ .

■

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In his manuscript, Johannsen made some observations concerning  $I\Delta_0$  provability that  $D_{m+1,k} \rightarrow D_{m,k}$ ; since it follows from this paper, that  $I\Delta_0$  proves the equivalence of the pigeonhole principle and Degen’s principle, we have answered Johannsen’s principal question. The resolution proofs were typeset using  $\LaTeX$ macros developed by S. Buss.

## References

- [1] R. Boppana and M. Sipser. The complexity of finite functions. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume A, pages 759–804. Elsevier, MIT Press, 1990. Elsevier (Amsterdam), MIT Press (Cambridge).
- [2] S.R. Buss and P. Clote. Cutting planes, connectivity, and threshold logic. *Archive for Mathematical Logic*, 35:33–62, 1966.
- [3] V. Chvátal. Edmonds polytopes and hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.
- [4] P. Clote. Note on monotonic complexity of 2 – REF. *Information Processing Letters*, 57:117–123, 1996.
- [5] P. Clote. Cutting plane and Frege proofs. *Information and Computation*, 121(1):103–122, 1995.
- [6] W. Cook, C.R. Coullard, and G. Turan. On the complexity of cutting plane proofs. *Discrete Applied Mathematics*, 18:25–38, 1987.
- [7] J.W. Degen. Pigeonhole principles and choice principles. Typescript, Universität Erlangen-Nürnberg, 1995.
- [8] A. Haken. The intractability of resolution. *Theoretical Computer Science*, 39:297 – 305, 1985.
- [9] A. Haken and S.A. Cook. An exponential lower bound for the size of monotonic real circuits. Typeset manuscript, December 23, 1995.
- [10] M. Karchmer and A. Wigderson. Monotone circuits for connectivity require super-logarithmic depth. *J. Discrete Mathematics*, 3, 1990. 255–265.
- [11] J. Krajíček. *Bounded Arithmetic, Propositional Logic, and Complexity Theory*. Cambridge University Press, 1995.
- [12] J. Krajíček. Interpolation theorems, lower bounds for proof systems and independence results for bounded arithmetic. Typeset manuscript, submitted, 1995.
- [13] P. Pudlák. Ramsey’s theorem in bounded arithmetic. In E. Börger, editor, *Proceedings of Computer Science Logic 1990*. Springer Lecture Notes in Computer Science **553**, 1992.
- [14] P. Pudlák. Lower bounds for resolution and cutting planes proofs and monotone computations. Typeset manuscript, 1995.
- [15] G. S. Tseitin. On the complexity of derivation in propositional calculus. In J. Siekmann and G. Wrigtson, editors, *Automation of Reasoning*, volume 2, pages 466 – 483. Springer Verlag, 1983.
- [16] A. Urquhart. Hard examples for resolution. *Journal of the Association of Computing Machinery*, 34(1):209 – 219, 1987.
- [17] A. Urquhart. The complexity of propositional proofs. *The Bulletin of Symbolic Logic*, 1:425–467, 1995.
- [18] I. Wegener. *Complexity of Boolean Functions*. Teubner-Wiley, 1987.